DOMAIN CLOSURE CONDITIONS AND DEFINABILITY PRESERVATION

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ABSTRACT

We show in Section (2) the importance of closure of the domain under finite unions, in particular for Cumulativity, and representation results. We see that in the absence of this closure, Cumulativity fans out to an infinity of different conditions.

We introduce in Section (3) the concept of an algebraic limit, and discuss its importance. We then present a representation result for a new concept of revision, introduced by Booth et al., using approximation by formulas.

We analyse in Section (4) definability preservation problems, and show that intersection is the crucial step. We simplify older proofs for the non-definability cases, and add a new result for ranked structures.

The (very brief) Section 5 puts our considerations in a larger perspective, and gives an outlook on open questions.

1 INTRODUCTION

1.1 Overview

We use and go beyond [Sch04] here. Thus, we will use some results shown there, and put them into a more general perspective. We will also re-prove some results of [Sch04] by more general means. And there are also a number of new results in the present text, which complement those of [Sch04], for instance, solving new, or more general cases.

The subject are closure conditions (and related themes) of the domain. We consider two types, first closure of the domain under simple set-theoretic operations, particularly under finite unions, second, whether the algebraic choice functions (or similar objects in the limit case) goes from definable sets to definable sets, or to arbitrary subsets, i.e. whether the domain is closed under these functions, in other words whether these functions preserve definability.

In particular, we will show that the absence of closure under finite unions has important consequences for a property called cumulativity. Without this closure, there is an infinity of different versions of cumulativity, which all collaps to one condition in the presence of closure.

This is motivated by the following: The sets of theory definable propositional model sets - i.e. of the type $M(T) := \{m : m \models T\}$ - are, in the infinite case, closed under arbitrary intersections, finite unions, but not set difference. When we consider logics defined e.g. by suitable sequent calculi, closure under finite unions need not hold any longer. This has far reaching consequences for representation problems, as the author first noted in

[Leh92a]. We investigate the problem and solutions now systematically.

Domain closure problems interfere also in the limit variant of several logics, and tend to complicate the already relatively complex picture. As an example of the limit variant, we take preferential logics. $\mu(T)$ (the set of minimal models of T, considered in the "minimal variant") might well be empty, even if $M(T)$ is not empty, e.g. due to infinite $< -$descending chains of models. This leads to a degenerate case, which can be avoided by considering those formulas as consequences, which hold "finally" or "in the limit", i.e. from a certain level onward, even if there are no ideal (minimal) elements.

This limit approach (to various structures and logics) is particularly recalcitrant, as algebraic and definability preserving problems may occur together or separately. To distinguish between the two, we introduce the concept of an algebraic limit, and use this to re-prove as simple corollaries of more general situations previous trivialization results, i.e. cases where the more complicated limit case can be reduced to the simpler minimal case.

Thus, conceptually, we distinguish between the logical, the algebraic, and the structural situation (e.g., preferential structures create certain model choice functions, which create certain logics - we made this distinction already in our [Sch90] paper), and distinguish now also between the structural and the algebraic limit. We argue that a "good" limit should not only capture the idea of a structural limit, but it should also be an algebraic limit, i.e. capture the essential algebraic properties of the minimal choice functions. There might still be definability problems, but our approach allows to distinguish them now clearly. The latter, definability, problems are closure questions, common to the limit and the minimal variant, and can thus be treated together or in a similar way.

Note that these clear distinctions have some philosophical importance, too. The structures need an intuitive or philosophical justification, why do we describe preference by transitive relations, why do we admit copies, etc.? The resulting algebraic choice functions are void of such questions.

We summarize the distinction:

For e.g. preferential structures, we have:

- logical laws or descriptions like $\alpha \mid\!\sim \alpha$ - they are the (imperfect - by definability preservation problems) reflection of the abstract description,

- abstract or algebraic semantics, like $\mu(A) \subseteq A$ - they are the abstract description of the foundation,

- structural semantics - they are the intuitive foundation.

Likewise, for the limit situation, we have:

- structural limits - they are again the foundation,

- resulting abstract behaviour, which, again, has to be an abstract or algebraic limit, resulting from the structural limit,

- a logical limit, which reflects the abstract limit, and may be plagued by definability preservation problems etc. when going from the model to the logics side.

1.2 Basic definitions

We summarize now frequently used logical and algebraic properties in the following table. The left hand column presents the single formula version, the center column the theory version (a theory is, for us, an arbitrary set of formulas), the right hand column the algebraic version, describing the choice function on the model set, e.g. $f(X) \subseteq X$ corresponds to the rule $\phi \mid\sim \psi$ implies $\phi \mid\sim \psi$ in the formula version, and to $\overline{T} \subseteq \overline{\overline{T}}$ in the theory version. A short discussion of some of the properties follows the table.

Definition 1.1

(AND) $\phi \mid\!\sim \psi, \phi \mid\!\sim \psi' \Rightarrow$ $\phi \mid\!\sim \psi \wedge \psi'$	(AND) $T \mid\!\sim \psi, T \mid\!\sim \psi' \Rightarrow$ $T \mid\!\sim \psi \wedge \psi'$	
(OR) $\phi \mid\!\sim \psi, \phi' \mid\!\sim \psi \Rightarrow$ $\phi \vee \phi' \mid\!\sim \psi$	(OR) $T \mid\!\sim \psi, T' \mid\!\sim \psi \Rightarrow$ $T \vee T' \mid\!\sim \psi$	$(\mu \cup w)$ - w for weak $f(A \cup B) \subseteq f(A) \cup f(B)$
(LLE) or Left Logical Equivalence $\vdash \phi \leftrightarrow \phi', \phi \mid\!\sim \psi \Rightarrow$ $\phi' \mid\!\sim \psi$	(LLE) $\overline{T} = \overline{T'} \Rightarrow \overline{\overline{T}} = \overline{\overline{T'}}$	
(RW) or Right Weakening $\phi \mid\!\sim \psi, \vdash \psi \rightarrow \psi' \Rightarrow$ $\phi \mid\!\sim \psi'$	(RW) $T \mid\!\sim \psi, \vdash \psi \rightarrow \psi' \Rightarrow$ $T \mid\!\sim \psi'$	
(CCL) or Classical Closure	(CCL) $\overline{\overline{T}}$ is classically closed	
(SC) or Supraclassicality $\phi \vdash \psi \Rightarrow \phi \mid\!\sim \psi$	(SC) $\overline{T} \subseteq \overline{\overline{T}}$	$(\mu \subseteq)$ $f(X) \subseteq X$
(CP) or Consistency Preservation $\phi \mid\!\sim \bot \Rightarrow \phi \vdash \bot$	(CP) $T \mid\!\sim \bot \Rightarrow T \vdash \bot$	$(\mu\emptyset)$ $f(X) = \emptyset \Rightarrow X = \emptyset$
(RM) or Rational Monotony $\phi \mid\!\sim \psi, \phi \not\mid\!\sim \psi' \Rightarrow$ $\phi \wedge \psi' \mid\!\sim \psi$	(RM) $T \mid\!\sim \psi, T \not\mid\!\sim \psi' \Rightarrow$ $T \cup \{\psi'\} \mid\!\sim \psi$	$(\mu =)$ $X \subseteq Y, Y \cap f(X) \neq \emptyset \Rightarrow$ $f(X) = f(Y) \cap X$
(CM) or Cautious Monotony $\phi \mid\!\sim \psi, \phi \mid\!\sim \psi' \Rightarrow$ $\phi \wedge \psi \mid\!\sim \psi'$	(CM) $T \subseteq \overline{T'} \subseteq \overline{\overline{T}} \Rightarrow$ $\overline{\overline{T}} \subseteq \overline{\overline{T'}}$	$f(X) \subseteq Y \subseteq X \Rightarrow$ $f(Y) \subseteq f(X)$
(CUM) or Cumulativity $\phi \mid\!\sim \psi \Rightarrow$ $(\phi \mid\!\sim \psi' \Leftrightarrow \phi \wedge \psi \mid\!\sim \psi')$	(CUM) $T \subseteq \overline{T'} \subseteq \overline{\overline{T}} \Rightarrow$ $\overline{\overline{T}} = \overline{\overline{T'}}$	(μCUM) $f(X) \subseteq Y \subseteq X \Rightarrow$ $f(Y) = f(X)$
$\overline{\overline{\phi \wedge \phi'}} \subseteq \overline{\overline{\overline{\phi}} \cup \{\phi'\}}$	(PR) $\overline{\overline{T \cup T'}} \subseteq \overline{\overline{\overline{T}} \cup T'}$	(μPR) $X \subseteq Y \Rightarrow$ $f(Y) \cap X \subseteq f(X)$

(PR) is also called infinite conditionalization - we choose the name for its central role for preferential structures (PR) or (μPR). Note that in the presence of $(\mu \subseteq)$, and if $\mathcal{Y}$ is closed under finite intersections, (μPR) is equivalent to

$(\mu PR')$ $f(X) \cap Y \subseteq f(X \cap Y)$.

The system of rules (AND), (OR), (LLE), (RW), (SC), (CP), (CM), (CUM) is also called system P (for preferential), adding (RM) gives the system R (for rationality or rankedness).

(AND) is obviously closely related to filters, as we saw already in Section 1. (LLE), (RW), (CCL) will all hold automatically, whenever we work with fixed model sets. (SC) corresponds to the choice of a subset. (CP) is somewhat delicate, as it presupposes that the chosen model set is non-empty. This might fail in the presence of ever better choices, without ideal ones; the problem is addressed by the limit versions - see below in Section 3.4. (PR) is an inifinitary version of one half of the deduction theorem: Let T stand for ϕ, T' for ψ, and $\phi \wedge \psi \mid\sim \sigma$, so $\phi \mid\sim \psi \rightarrow \sigma$, but $(\psi \rightarrow \sigma) \wedge \psi \vdash \sigma$. (CUM) (whose most interesting half in our context is (CM)) may best be seen as normal use of lemmas: We have worked hard and found some lemmas. Now we can take a rest, and come back again with our new lemmas. Adding them to the axioms will neither add new theorems, nor prevent old ones to hold. (RM) is perhaps best understood by looking at big and small subsets. If the set of $\phi \wedge \psi -$models is a big subset of the set of $\phi -$models, and the set of $\phi \wedge \neg\psi' -$models is a not a small subset of the set of $\phi -$models (i.e. big or of medium size), then the set of $\phi \wedge \psi \wedge \psi' -$models is a big subset of the set of $\phi \wedge \psi' -$models.

The following two definitions make preferential structures precise. We first give the algebraic definition, and then the definition of the consequence relation generated by an preferential structure. In the algebraic definition, the set U is an arbitrary set, in the application to logic, this will be the set of classical models of the underlying propositional language.

In both cases, we first present the simpler variant without copies, and then the one with copies. (Note that e.g. [KLM90], [LM92] use labelling functions instead, the version without copies corresponds to injective labelling functions, the one with copies to the general case. These are just different ways of speaking.) We will discuss the difference between the version without and the version with copies below, where we show that the version with copies is strictly more expressive than the version without copies, and that transitivity of the relation adds new properties in the case without copies. When we summarize our own results below (see Section 2.2.3), we will mention that, in the general case with copies, transitivity can be added without changing properties.

We give here the "minimal version", the much more complicated "limit version" is presented and discussed in Section 3. Recall the intuition that the relation $\prec$ expresses "normality" or "importance" - the $\prec -$smaller, the more normal or important. The smallest elements are those which count.

Definition 1.2

Fix $U \neq \emptyset$, and consider arbitrary X. Note that this X has not necessarily anything to do with U, or $\mathcal{U}$ below. Thus, the functions $\mu_{\mathcal{M}}$ below are in principle functions from V to V - where V is the set theoretical universe we work in.

(A) Preferential models or structures.

(1) The version without copies:

A pair $\mathcal{M} :=< U, \prec>$ with U an arbitrary set, and $\prec$ an arbitrary binary relation is called a preferential model or structure.

(2) The version with copies:

A pair $\mathcal{M} :=< \mathcal{U}, \prec>$ with $\mathcal{U}$ an arbitrary set of pairs, and $\prec$ an arbitrary binary relation is called a preferential model or structure.

If $< x, i > \in \mathcal{U}$, then x is intended to be an element of U, and i the index of the copy.

(B) Minimal elements, the functions $\mu_{\mathcal{M}}$

(1) The version without copies:

Let $\mathcal{M} :=< U, \prec>$, and define

$\mu_{\mathcal{M}}(X) := \{x \in X : x \in U \wedge \neg \exists x' \in X \cap U . x' \prec x\}$.

$\mu_{\mathcal{M}}(X)$ is called the set of minimal elements of X (in $\mathcal{M}$).

(2) The version with copies:

Let $\mathcal{M} :=< \mathcal{U}, \prec>$ be as above. Define

$\mu_{\mathcal{M}}(X) := \{x \in X : \exists < x, i > \in \mathcal{U} . \neg \exists < x', i' > \in \mathcal{U}(x' \in X \wedge < x', i' >' \prec < x, i >)\}$.

Again, by abuse of language, we say that $\mu_{\mathcal{M}}(X)$ is the set of minimal elements of X in the structure. If the context is clear, we will also write just μ.

We sometimes say that $< x, i >$ "kills" or "minimizes" $< y, j >$ if $< x, i > \prec < y, j >$. By abuse of language we also say a set X kills or minimizes a set Y if for all $< y, j > \in \mathcal{U}$, $y \in Y$ there is $< x, i > \in \mathcal{U}$, $x \in X$ s.t. $< x, i > \prec < y, j >$.

$\mathcal{M}$ is also called injective or 1-copy, iff there is always at most one copy $< x, i >$ for each x.

We say that $\mathcal{M}$ is transitive, irreflexive, etc., iff $\prec$ is.

Recall that $\mu(X)$ might well be empty, even if X is not.

Definition 1.3

We define the consequence relation of a preferential structure for a given propositional language $\mathcal{L}$.

(A)

(1) If m is a classical model of a language $\mathcal{L}$, we say by abuse of language

$< m, i > \models \phi$ iff $m \models \phi$,

and if X is a set of such pairs, that

$X \models \phi$ iff for all $< m, i > \in X$ $m \models \phi$.

(2) If $\mathcal{M}$ is a preferential structure, and X is a set of $\mathcal{L}-$models for a classical propositional language $\mathcal{L}$, or a set of pairs $< m, i >$, where the m are such models, we call $\mathcal{M}$ a classical preferential structure or model.

(B)

Validity in a preferential structure, or the semantical consequence relation defined by such a structure:

Let $\mathcal{M}$ be as above.

We define:

$T \models_{\mathcal{M}} \phi$ iff $\mu_{\mathcal{M}}(M(T)) \models \phi$, i.e. $\mu_{\mathcal{M}}(M(T)) \subseteq M(\phi)$.

$\mathcal{M}$ will be called definability preserving iff for all $X \in \boldsymbol{D}_{\mathcal{L}}$ $\mu_{\mathcal{M}}(X) \in \boldsymbol{D}_{\mathcal{L}}$.

As $\mu_{\mathcal{M}}$ is defined on $\boldsymbol{D}_{\mathcal{L}}$, but need by no means always result in some new definable set, this is (and reveals itself as a quite strong) additional property.

We define now two additional properties of the relation, smoothness and rankedness.

The first condition says that if $x \in X$ is not a minimal element of X, then there is $x' \in \mu(X)$ $x' \prec x$. In the finite case without copies, smoothness is a trivial consequence of transitivity and lack of cycles. But note that in the other cases infinite descending chains might still exist, even if the smoothness condition holds, they are just "short-circuited": we might have such chains, but below every element in the chain is a minimal element. In the author's opinion, smoothness is difficult to justify as a structural property (or, in a more philosophical spirit, as a property of the world): why should we always have such minimal elements below non-minimal ones? Smoothness has, however, a justification from its consequences. Its attractiveness comes from two sides:

First, it generates a very valuable logical property, cumulativity (CUM): If $\mathcal{M}$ is smooth, and $\overline{\overline{T}}$ is the set of $\models_{\mathcal{M}}$-consequences, then for $T \subseteq \overline{T'} \subseteq \overline{\overline{T}} \Rightarrow \overline{\overline{T}} = \overline{\overline{T'}}$.

Second, for certain approaches, it facilitates completeness proofs, as we can look directly at "ideal" elements, without having to bother about intermediate stages. See in particular the work by Lehmann and his co-authors, [KLM90], [LM92].

"Smoothness", or, as it is also called, "stopperedness" seems - in the author's opinion - a misnamer. *I* think it should better be called something like "weak transitivity": consider the case where $a \succ b \succ c$, but $c \not\prec a$, with $c \in \mu(X)$. It is then not necessarily the case that $a \succ c$, but there is c' "sufficiently close to c", i.e. in $\mu(X)$, s.t. $a \succ c'$. Results and proof techniques underline this idea. First, in the general case with copies, and in the smooth case (in the presence of $(\cup)$!), transitivity does not add new properties, it is "already present", second, the construction of smoothness by sequences σ (see below in Section 2.3) is very close in spirit to a transitive construction.

The second condition, rankedness, seems easier to justify already as a property of the structure. It says that, essentially, the elements are ordered in layers: If a and b are not comparable, then they are in the same layer. So, if c is above (below) a, it will also be above (below) b - like pancakes or geological strata. Apart from the triangle inequality (and leaving aside cardinality questions), this is then just a distance from some imaginary, ideal point. Again, this property has important consequences on the resulting model choice functions and consequence relations, making proof techniques for the non-ranked and the ranked case very different.

Definition 1.4

Let $\mathcal{Z} \subseteq \mathcal{P}(U)$. (In applications to logic, $\mathcal{Z}$ will be $\boldsymbol{D}_{\mathcal{L}}$.)

A preferential structure $\mathcal{M}$ is called $\mathcal{Z}-$smooth iff in every $X \in \mathcal{Z}$ every element $x \in X$ is either minimal in X or above an element, which is minimal in X. More precisely:

(1) The version without copies:

If $x \in X \in \mathcal{Z}$, then either $x \in \mu(X)$ or there is $x' \in \mu(X).x' \prec x$.

(2) The version with copies:

If $x \in X \in \mathcal{Z}$, and $< x, i > \in \mathcal{U}$, then either there is no $< x', i' > \in \mathcal{U}$, $x' \in X$, $< x', i' > \prec < x, i >$ or there is $< x', i' > \in \mathcal{U}$, $< x', i' > \prec < x, i >$, $x' \in X$, s.t. there is no $< x'', i'' > \in \mathcal{U}$, $x'' \in X$, with $< x'', i'' > \prec < x', i' >$.

When considering the models of a language $\mathcal{L}$, $\mathcal{M}$ will be called smooth iff it is $\boldsymbol{D}_{\mathcal{L}}-$smooth; $\boldsymbol{D}_{\mathcal{L}}$ is the default.

Obviously, the richer the set $\mathcal{Z}$ is, the stronger the condition $\mathcal{Z}-$smoothness will be.

Definition 1.5

A relation $\prec_U$ on U is called ranked iff there is an order-preserving function from U to a total order O, $f : U \to O$, with $u \prec_U u'$ iff $f(u) \prec_O f(u')$, equivalently, if x and x' are $\prec_U$ $-$incomparable,

then ($y \prec_U x$ iff $y \prec_U x'$) and ($y \succ_U x$ iff $y \succ_U x'$) for all y.

We conclude with the following standard example:

Example 1.1

If $v(\mathcal{L})$ is infinite, and m any model for $\mathcal{L}$, then $M := M_{\mathcal{L}} - \{m\}$ is not definable by any theory T. (Proof: Suppose it were, and let ϕ hold in M', but not in m, so in m $\neg\phi$ holds, but as ϕ is finite, there is a model m' in M' which coincides on all propositional variables of ϕ with m, so in m' $\neg\phi$ holds, too, a contradiction.)

2 UNIONS AND CUMULATIVITY

2.1 Introduction

This section was motivated by Lehmann's Plausibility Logic, and re-motivated by the work of Arieli and Avron, see [AA00]. In both cases, the language does not have a built-in "or" - resulting in absence $(\cup)$ of the domain. It is thus an essay on the enormous strength of closure of the domain under finite unions, and, more generally, on the importance of domain closure conditions.

In the resulting completeness proofs again, a strategy of "divide and conquer" is useful. This helps us to unify (or extend) our past completeness proofs for the smooth case in the following way: We will identify more clearly than in the past a more or less simple algebraic property - (HU), (HUx) etc. - which allows us to split the proofs into two parts. The first part shows validity of the property, and this demonstration depends on closure properties, the second part shows how to construct a representing structure using the algebraic property. This second part will be totally independent from closure properties, and is essentially an "administrative" way to use the property for a construction. This split approach allows us thus to isolate the demonstration of the used property from the construction itself, bringing both parts clearer to light, and simplifying the proofs, by using common parts.

The reader will see that the successively more complicated conditions (HU), (HUx), $(\mu\tau)$ reflect well the successively more complicated situations of representation:

(HU): smooth (and transitive) structures in the presence of $(\cup)$,

(HUx): smooth structures in the absence of $(\cup)$,

$(\mu\tau)$: smooth and transitive structures in the absence of $(\cup)$.

This comparison becomes clearer when we see that in the final, most complicated case, we will have to carry around all the history of minimization, $< Y_0, \ldots, Y_n >$, necessary for transitivity, which could be summarized in the first case with to finite unions. Thus, from an abstract point of view, it is a very natural development.

2.2 Basic definitions and results

We show that, without sufficient closure properties, there is an infinity of versions of cumulativity, which collaps to usual cumulativity when the domain is closed under finite unions. Closure properties thus reveal themselves as a powerful tool to show independence of properties.

We work in some fixed arbitrary set Z, all sets considered will be subsets of Z.

We recall or define

Definition 2.1

(μPR) $X \subseteq Y \rightarrow \mu(Y) \cap X \subseteq \mu(X)$

$(\mu \subseteq)$ $\mu(X) \subseteq X$

(μCum) $\mu(X) \subseteq Y \subseteq X \rightarrow \mu(X) = \mu(Y)$

$(\cup)$ is closure of the domain under finite unions.

$(\cap)$ is closure of the domain under finite intersections.

$(\bigcap)$ is closure of the domain under arbitrary intersections.

We use without further mentioning (μPR) and $(\mu \subseteq)$.

Definition 2.2

For any ordinal α, we define

$(\mu Cum\alpha)$:

If for all $\beta \leq \alpha$ $\mu(X_\beta) \subseteq U \cup \bigcup\{X_\gamma : \gamma < \beta\}$ hold, then so does $\bigcap\{X_\gamma : \gamma \leq \alpha\} \cap \mu(U) \subseteq \mu(X_\alpha)$.

$(\mu Cumt\alpha)$:

If for all $\beta \leq \alpha$ $\mu(X_\beta) \subseteq U \cup \bigcup\{X_\gamma : \gamma < \beta\}$ hold, then so does $X_\alpha \cap \mu(U) \subseteq \mu(X_\alpha)$.

("t" stands for transitive, see Fact 2.1, (2.2) below.)

$(\mu Cum\infty)$ and $(\mu Cumt\infty)$ will be the class of all $(\mu Cum\alpha)$ or $(\mu Cumt\alpha)$ - read their "conjunction", i.e. if we say that $(\mu Cum\infty)$ holds, we mean that all $(\mu Cum\alpha)$ hold.

The following inductive definition of $H(U, x)$ and of the property (HUx) concerns closure under $(\mu Cum\infty)$, its main property is formulated in Fact 2.1, (8), its main interest is its use in the proof of Proposition 2.2.

$H(U, x)_0 := U$,

$H(U, x)_{\alpha+1} := H(U, x)_\alpha \cup \bigcup\{X : x \in X \wedge \mu(X) \subseteq H(U, x)_\alpha\}$,

$H(U, x)_\lambda := \bigcup\{H(U, x)_\alpha : \alpha < \lambda\}$ for $limit(\lambda)$,

$H(U, x) := \bigcup\{H(U, x)_\alpha : \alpha < \kappa\}$ for κ sufficiently big ($card(Z)$ suffices, as

the procedure trivializes, when we cannot add any new elements).

(HUx) is the property:

$x \in \mu(U)$, $x \in Y - \mu(Y) \rightarrow \mu(Y) \not\subseteq H(U, x)$ - of course for all x and U. $(U, Y \in \mathcal{Y})$.

For the case with $(\cup)$, we further define, independent of x,

$H(U)_0 := U$,

$H(U)_{\alpha+1} := H(U)_\alpha \cup \bigcup\{X : \mu(X) \subseteq H(U)_\alpha\}$,

$H(U)_\lambda := \bigcup\{H(U)_\alpha : \alpha < \lambda\}$ for $limit(\lambda)$,

$H(U) := \bigcup\{H(U)_\alpha : \alpha < \kappa\}$ again for κ sufficiently big

(HU) is the property:

$x \in \mu(U)$, $x \in Y - \mu(Y) \to \mu(Y) \not\subseteq H(U)$ - of course for all U. $(U, Y \in \mathcal{Y})$.

Obviously, $H(U,x) \subseteq H(U)$, so $(HU) \to (HUx)$.

Note:

The first conditions thus have the form:

$(\mu Cum0)$ $\mu(X_0) \subseteq U \to X_0 \cap \mu(U) \subseteq \mu(X_0)$,

$(\mu Cum1)$ $\mu(X_0) \subseteq U$, $\mu(X_1) \subseteq U \cup X_0 \to X_0 \cap X_1 \cap \mu(U) \subseteq \mu(X_1)$,

$(\mu Cum2)$ $\mu(X_0) \subseteq U$, $\mu(X_1) \subseteq U \cup X_0$, $\mu(X_2) \subseteq U \cup X_0 \cup X_1 \to X_0 \cap X_1 \cap X_2 \cap \mu(U) \subseteq \mu(X_2)$.

$(\mu Cumt\alpha)$ differs from $(\mu Cum\alpha)$ only in the consequence, the intersection contains only the last X_α - in particular, $(\mu Cum0)$ and $(\mu Cumt0)$ coincide.

Recall that condition $(\mu Cum1)$ is the crucial condition in [Leh92a], which failed, despite (μCum), but which has to hold in all smooth models. This condition $(\mu Cum1)$ was the starting point of the investigation.

Example 2.1

Perhaps the main result of this section is the following example, which shows that above conditions are all different in the absence of $(\cup)$, in its presence they all collaps (see .... below). More precisely, the following (class of) *example(s)* shows that the $(\mu Cum\alpha)$ increase in strength. For any finite or infinite ordinal $\kappa > 0$ we construct an example s.t.

(a) (μPR) and $(\mu \subseteq)$ hold

(b) (μCum) holds

(c) $(\bigcap)$ holds

(d) $(\mu Cumt\alpha)$ holds for $\alpha < \kappa$

(e) $(\mu Cum\kappa)$ fails.

We define a suitable base set and a non-transitive binary relation $\prec$ on this set, as well as a suitable set $\mathcal{X}$ of subsets, closed under arbitrary intersections, but not under finite unions, and define μ on these subsets as usual in preferential structures by $\prec$. Thus, (μPR) and $(\mu \subseteq)$ will hold. It will be immediate that $(\mu Cum\kappa)$ fails, and we will show that (μCum) and $(\mu Cumt\alpha)$ for $\alpha < \kappa$ hold by examining the cases.

For simplicity, we first define a set of generators for $\mathcal{X}$, and close under $(\bigcap)$ afterwards. The set U will have a special position, it is the "useful" starting point to construct chains corresponding to above definitions of $(\mu Cum\alpha)$ and $(\mu Cumt\alpha)$.

Notation 2.1

i,j will be successor ordinals, λ etc. limit ordinals, α, β, κ any ordinals, thus e.g. $\lambda \leq \kappa$ will imply that λ is a limit ordinal $\leq \kappa$, etc.

The base set and the relation $\prec$:

$\kappa > 0$ is fixed, but arbitrary. We go up to $\kappa > 0$.

The base set is $\{a, b, c\} \cup \{d_\lambda : \lambda \leq \kappa\} \cup \{x_\alpha : \alpha \leq \kappa + 1\} \cup \{x'_\alpha : \alpha \leq \kappa\}$. $a \prec b \prec c$, $x_\alpha \prec x_{\alpha+1}$, $x_\alpha \prec x'_\alpha$, $x'_0 \prec x_\lambda$ (for any λ) - $\prec$ is NOT transitive.

The generators:

$U := \{a, c, x_0\} \cup \{d_\lambda : \lambda \leq \kappa\}$ - i.e. $\ldots.\{d_\lambda : lim(\lambda) \wedge \lambda \leq \kappa\}$,

$X_i := \{c, x_i, x'_i, x_{i+1}\}$ $(i < \kappa)$,

$X_\lambda := \{c, d_\lambda, x_\lambda, x'_\lambda, x_{\lambda+1}\} \cup \{x'_\alpha : \alpha < \lambda\}$ $(\lambda < \kappa)$,

$X'_\kappa := \{a, b, c, x_\kappa, x'_\kappa, x_{\kappa+1}\}$ if κ is a successor,

$X'_\kappa := \{a, b, c, d_\kappa, x_\kappa, x'_\kappa, x_{\kappa+1}\} \cup \{x'_\alpha : \alpha < \kappa\}$ if κ is a limit.

Thus, $X'_\kappa = X_\kappa \cup \{a, b\}$ if X_κ were defined.

Note that there is only one X'_κ, and X_α is defined only for $\alpha < \kappa$, so we will not have X_α and X'_α at the same time.

Thus, the values of the generators under μ are:

$\mu(U) = U$,

$\mu(X_i) = \{c, x_i\}$,

$\mu(X_\lambda) = \{c, d_\lambda\} \cup \{x'_\alpha : \alpha < \lambda\}$,

$\mu(X'_i) = \{a, x_i\}$ $(i > 0!)$,

$\mu(X'_\lambda) = \{a, d_\lambda\} \cup \{x'_\alpha : \alpha < \lambda\}$.

(We do not assume that the domain is closed under μ.)

Intersections:

We consider first pairwise intersections:

(1) $U \cap X_0 = \{c, x_0\}$,

(2) $U \cap X_i = \{c\}$, $i > 0$,

(3) $U \cap X_\lambda = \{c, d_\lambda\}$,

(4) $U \cap X'_i = \{a, c\}$ $(i > 0!)$,

(5) $U \cap X'_\lambda = \{a, c, d_\lambda\}$,

(6) $X_i \cap X_j :$

(6.1) $j = i + 1$ $\{c, x_{i+1}\}$,

(6.2) else $\{c\}$,

(7) $X_i \cap X_\lambda :$

(7.1) $i < \lambda$ $\{c, x'_i\}$,

(7.2) $i = \lambda + 1$ $\{c, x_{\lambda+1}\}$,

(7.3) $i > \lambda + 1$ $\{c\}$,

(8) $X_\lambda \cap X_{\lambda'} : \{c\} \cup \{x'_\alpha : \alpha \leq min(\lambda, \lambda')\}$.

As X'_κ occurs only once, $X_\alpha \cap X'_\kappa$ etc. give no new results.

Note that μ is constant on all these pairwise intersections.

Iterated intersections:

As c is an element of all sets, sets of the type $\{c, z\}$ do not give any new results. The possible subsets of $\{a, c, d_\lambda\}$: $\{c\}$, $\{a, c\}$, $\{c, d_\lambda\}$ exist already. Thus, the only source of new sets via iterated intersections is $X_\lambda \cap X_{\lambda'} = \{c\} \cup \{x'_\alpha : \alpha \leq min(\lambda, \lambda')\}$. But, to intersect them, or with some old sets, will not generate any new sets either. Consequently, the example satisfies $(\bigcap)$ for $\mathcal{X}$ defined by U, X_i $(i < \kappa)$, X_λ $(\lambda < \kappa)$, X'_κ, and above paiwise intersections.

We will now verify the positive properties. This is tedious, but straightforward, checking the different cases.

Validity of (μCum)**:**

Consider the prerequisite $\mu(X) \subseteq Y \subseteq X$. If $\mu(X) = X$ or if $X - \mu(X)$ is a singleton, X cannot give a violation of (μCum). So we are left with the following candidates for X :

(1) $X_i := \{c, x_i, x'_i, x_{i+1}\}$, $\mu(X_i) = \{c, x_i\}$

Interesting candidates for Y will have 3 elements, but they will all contain a. (If $\kappa < \omega$: $U = \{a, c, x_0\}$.)

(2) $X_\lambda := \{c, d_\lambda, x_\lambda, x'_\lambda, x_{\lambda+1}\} \cup \{x'_\alpha : \alpha < \lambda\}$, $\mu(X_\lambda) = \{c, d_\lambda\} \cup \{x'_\alpha : \alpha < \lambda\}$

The only sets to contain d_λ are X_λ, U, $U \cap X_\lambda$. But $a \in U$, and $U \cap X_\lambda$ ist finite. (X_λ and X'_λ cannot be present at the same time.)

(3) $X'_i := \{a, b, c, x_i, x'_i, x_{i+1}\}$, $\mu(X'_i) = \{a, x_i\}$

a is only in U, X'_i, $U \cap X'_i = \{a, c\}$, but $x_i \notin U$, as $i > 0$.

(4) $X'_\lambda := \{a, b, c, d_\lambda, x_\lambda, x'_\lambda, x_{\lambda+1}\} \cup \{x'_\alpha : \alpha < \lambda\}$, $\mu(X'_\lambda) = \{a, d_\lambda\} \cup \{x'_\alpha : \alpha < \lambda\}$

d_λ is only in X'_λ and U, but U contains no x'_α.

Thus, (μCum) holds trivially.

$(\mu Cumt\alpha)$ **hold for** $\alpha < \kappa$**:**

To simplify language, we say that we reach Y from X iff $X \neq Y$ and there is a sequence X_β, $\beta \leq \alpha$ and $\mu(X_\beta) \subseteq X \cup \bigcup\{X_\gamma : \gamma < \beta\}$, and $X_\alpha = Y$, $X_0 = X$. Failure of $(\mu Cumt\alpha)$ would then mean that there are X and Y, we can reach Y from X, and $x \in (\mu(X) \cap Y) - \mu(Y)$. Thus, in a counterexample, $Y = \mu(Y)$ is impossible, so none of the intersections can be such Y.

To reach Y from X, we have to get started from X, i.e. there must be Z s.t. $\mu(Z) \subseteq X$, $Z \not\subseteq X$ (so $\mu(Z) \neq Z$). Inspection of the different cases shows that we cannot reach any set Y from any case of the intersections, except from (1), (6.1), (7.2).

If Y contains a globally minimal element (i.e. there is no smaller element in any set), it can only be reached from any X which already contains this element. The globally minimal elements are a, x_0, and the d_λ, $\lambda \leq \kappa$.

By these observations, we see that X_λ and X'_κ can only be reached from U. From no X_α U can be reached, as the globally minimal a is missing. But U cannot be reached from X'_κ either, as the globally minimal x_0 is missing.

When we look at the relation $\prec$ defining μ, we see that we can reach Y from X only by going upwards, adding bigger elements. Thus, from X_α, we cannot reach any X_β, $\beta < \alpha$, the same holds for X'_κ and X_β, $\beta < \kappa$. Thus, from X'_κ, we cannot go anywhere interesting (recall that the intersections are not candidates for a Y giving a contradiction).

Consider now X_α. We can go up to any $X_{\alpha+n}$, but not to any X_λ, $\alpha < \lambda$, as d_λ is missing, neither to X'_κ, as a is missing. And we will be stopped by the first $\lambda > \alpha$, as x_λ will be missing to go beyond X_λ. Analogous observations hold for the remaining intersections (1), (6.1), (7.2). But in all these sets we can reach, we will not destroy minimality of any element of X_α (or of the intersections).

Consequently, the only candidates for failure will all start with U. As the only element of U not globally minimal is c, such failure has to have $c \in Y - \mu(Y)$, so Y has to be X'_κ. Suppose we omit one of the X_α in the sequence going up to X'_κ. If $\kappa \geq \lambda > \alpha$, we cannot reach X_λ and beyond, as x'_α will be missing. But we cannot go to $X_{\alpha+n}$ either, as $x_{\alpha+1}$ is missing. So we will be stopped at X_α. Thus, to see failure, we need the full sequence $U = X_0$, $X'_\kappa = Y_\kappa$, $Y_\alpha = X_\alpha$ for $0 < \alpha < \kappa$.

$(\mu Cum\kappa)$ **fails:**

The full sequence $U = X_0$, $X'_\kappa = Y_\kappa$, $Y_\alpha = X_\alpha$ for $0 < \alpha < \kappa$ shows this, as $c \in \mu(U) \cap X'_\kappa$, but $c \notin \mu(X'_\kappa)$.

Consequently, the example satisfies $(\bigcap)$, (μCum), $(\mu Cumt\alpha)$ for $\alpha < \kappa$, and $(\mu Cum\kappa)$ fails.

□

Fact 2.1

We summarize some properties of $(\mu Cum\alpha)$ and $(\mu Cumt\alpha)$ - sometimes with some redundancy. Unless said otherwise, α, β etc. will be arbitrary ordinals.

For (1) to (6) (μPR) and $(\mu \subseteq)$ are assumed to hold, for (7) and (8) only $(\mu \subseteq)$.

(1) Downward:

(1.1) $(\mu Cum\alpha) \rightarrow (\mu Cum\beta)$ for all $\beta \leq \alpha$

(1.2) $(\mu Cumt\alpha) \rightarrow (\mu Cumt\beta)$ for all $\beta \leq \alpha$

(2) Validity of $(\mu Cum\alpha)$ and $(\mu Cumt\alpha)$:

(2.1) All $(\mu Cum\alpha)$ hold in smooth preferential structures

(2.2) All $(\mu Cumt\alpha)$ hold in transitive smooth preferential structures

(2.3) $(\mu Cumt\alpha)$ for $0 < \alpha$ do not necessarily hold in smooth structures without transitivity, even in the presence of $(\bigcap)$

(3) Upward:

(3.1) $(\mu Cum\beta) + (\cup) \rightarrow (\mu Cum\alpha)$ for all $\beta \leq \alpha$

(3.2) $(\mu Cumt\beta) + (\cup) \rightarrow (\mu Cumt\alpha)$ for all $\beta \leq \alpha$

(3.3) $\{(\mu Cumt\beta) : \beta < \alpha\} + (\mu Cum) + (\bigcap) \not\rightarrow (\mu Cum\alpha)$ for $\alpha > 0$.

(4) Connection $(\mu Cum\alpha)/(\mu Cumt\alpha)$:

(4.1) $(\mu Cumt\alpha) \rightarrow (\mu Cum\alpha)$

(4.2) $(\mu Cum\alpha) + (\bigcap) \not\rightarrow (\mu Cumt\alpha)$

(4.3) $(\mu Cum\alpha) + (\cup) \rightarrow (\mu Cumt\alpha)$

(5) (μCum) and $(\mu Cumi)$:

(5.1) (μCum) + $(\cup)$ entail:

(5.1.1) $\mu(A) \subseteq B \to \mu(A \cup B) = \mu(B)$

(5.1.2) $\mu(X) \subseteq U, U \subseteq Y \to \mu(Y \cup X) = \mu(Y)$

(5.1.3) $\mu(X) \subseteq U, U \subseteq Y \to \mu(Y) \cap X \subseteq \mu(U)$

(5.2) $(\mu Cum\alpha) \to (\mu Cum)$ for all α

(5.3) (μCum) + $(\cup) \to (\mu Cum\alpha)$ for all α

(5.4) (μCum) + $(\cap) \to (\mu Cum0)$

(6) (μCum) and $(\mu Cumt\alpha)$:

(6.1) $(\mu Cumt\alpha) \to (\mu Cum)$ for all α

(6.2) (μCum) + $(\cup) \to (\mu Cumt\alpha)$ for all α

(6.3) $(\mu Cum) \not\to (\mu Cumt\alpha)$ for all $\alpha > 0$

(7) $(\mu Cum0) \to (\mu PR)$

(8) $(\mu Cum\infty)$ and (HUx):

(8.1) $x \in \mu(Y)$, $\mu(Y) \subseteq H(U,x) \to Y \subseteq H(U,x)$

(8.2) $(\mu Cum\infty) \to$ (HUx)

(8.3) (HUx) $\to (\mu Cum\infty)$

Proof:

We prove these facts in a different order: (1), (2), (5.1), (5.2), (4.1), (6.1), (6.2), (5.3), (3.1), (3.2), (4.2), (4.3), (5.4), (3.3), (6.3), (7), (8).

(1.1)

For $\beta < \gamma \leq \alpha$ set $X_\gamma := X_\beta$. Let the prerequisites of $(\mu Cum\beta)$ hold. Then for γ with $\beta < \gamma \leq \alpha$ $\mu(X_\gamma) \subseteq X_\beta$ by $(\mu \subseteq)$, so the prerequisites of $(\mu Cum\alpha)$ hold, too, so by $(\mu Cum\alpha)$ $\bigcap\{X_\delta : \delta \leq \beta\} \cap \mu(U) = \bigcap\{X_\delta : \delta \leq \alpha\} \cap \mu(U) \subseteq \mu(X_\alpha) = \mu(X_\beta)$.

(1.2)

Analogous.

(2.1)

Proof by induction.

$(\mu Cum0)$ Let $\mu(X_0) \subseteq U$, suppose there is $x \in \mu(U) \cap (X_0 - \mu(X_0))$. By smoothness, there is $y \prec x$, $y \in \mu(X_0) \subseteq U$, *contradiction* (The same arguments works for copies: all copies of x must be minimized by some $y \in \mu(X_0)$, but at least one copy of x has to be minimal in U.)

Suppose $(\mu Cum\beta)$ hold for all $\beta < \alpha$. We show $(\mu Cum\alpha)$. Let the prerequisites of $(\mu Cum\alpha)$ hold, then those for $(\mu Cum\beta)$, $\beta < \alpha$ hold, too. Suppose there is $x \in \mu(U) \cap \bigcap\{X_\gamma : \gamma \leq \alpha\} - \mu(X_\alpha)$. So by $(\mu Cum\beta)$ for $\beta < \alpha$ $x \in \mu(X_\beta)$ moreover $x \in \mu(U)$. By smoothness, there is $y \in \mu(X_\alpha) \subseteq U \cup \bigcup\{X_\beta : \beta < \alpha\}$, $y \prec x$, but this is a contradiction. The same argument works again for copies.

(2.2)

We use the following Fact: Let, in a smooth transitive structure, $\mu(X_\beta) \subseteq U \cup \bigcup\{X_\gamma : \gamma < \beta\}$ for all $\beta \leq \alpha$, and let $x \in \mu(U)$. Then there is no $y \prec x$, $y \in U \cup \bigcup\{X_\gamma : \gamma \leq \alpha\}$.

Proof of the Fact by induction: $\alpha = 0 : y \in U$ is impossible: if $y \in X_0$, then if $y \in \mu(X_0) \subseteq U$, which is impossible, or there is $z \in \mu(X_0)$, $z \prec y$, so $z \prec x$ by transitivity, but $\mu(X_0) \subseteq U$. Let the result hold for all $\beta < \alpha$, but fail for α, so $\neg\exists y \prec x.y \in U \cup \bigcup\{X_\gamma : \gamma < \alpha\}$, but $\exists y \prec x.y \in U \cup \bigcup\{X_\gamma : \gamma \leq \alpha\}$, so $y \in X_\alpha$. If $y \in \mu(X_\alpha)$, then $y \in U \cup \bigcup\{X_\gamma : \gamma < \alpha\}$, but this is impossible, so $y \in X_\alpha - \mu(X_\alpha)$, let by smoothness $z \prec y$, $z \in \mu(X_\alpha)$, so by transitivity $z \prec x$, *contradiction*. The result is easily modified for the case with copies.

Let the prerequisites of $(\mu Cumt\alpha)$ hold, then those of the Fact will hold, too. Let now $x \in \mu(U) \cap (X_\alpha - \mu(X_\alpha))$, by smoothness, there must be $y \prec x$, $y \in \mu(X_\alpha) \subseteq U \cup \bigcup\{X_\gamma : \gamma < \alpha\}$, contradicting the Fact.

(2.3)

Let $\alpha > 0$, and consider the following structure over $\{a, b, c\} : U := \{a, c\}$, $X_0 := \{b, c\}$, $X_\alpha := \ldots := X_1 := \{a, b\}$, and their intersections, $\{a\}$, $\{b\}$, $\{c\}$, $\emptyset$ with the order $c \prec b \prec a$ (without transitivity). This is preferential, so (μPR) and $(\mu \subseteq)$ hold. The structure is smooth for U, all X_β, and their intersections. We have $\mu(X_0) \subseteq U$, $\mu(X_\beta) \subseteq U \cup X_0$ for all $\beta \leq \alpha$, so $\mu(X_\beta) \subseteq U \cup \bigcup\{X_\gamma : \gamma < \beta\}$ for all $\beta \leq \alpha$ but $X_\alpha \cap \mu(U) = \{a\} \not\subseteq \{b\} = \mu(X_\alpha)$ for $\alpha > 0$.

(5.1)

(5.1.1) $\mu(A) \subseteq B \rightarrow \mu(A \cup B) \subseteq \mu(A) \cup \mu(B) \subseteq B \rightarrow_{(\mu Cum)} \mu(B) = \mu(A \cup B)$.

(5.1.2) $\mu(X) \subseteq U \subseteq Y \rightarrow$ (by (1)) $\mu(Y \cup X) = \mu(Y)$.

(5.1.3) $\mu(Y) \cap X =$ (by (2)) $\mu(Y \cup X) \cap X \subseteq \mu(Y \cup X) \cap (X \cup U) \subseteq$ (by (μPR)) $\mu(X \cup U)$ $=$ (by (1)) $\mu(U)$.

(5.2)

Using (1.1), it suffices to show $(\mu Cum0) \rightarrow (\mu Cum)$. Let $\mu(X) \subseteq U \subseteq X$. By $(\mu Cum0)$ $X \cap \mu(U) \subseteq \mu(X)$, so by $\mu(U) \subseteq U \subseteq X \rightarrow \mu(U) \subseteq \mu(X)$. $U \subseteq X \rightarrow \mu(X) \cap U \subseteq \mu(U)$, but also $\mu(X) \subseteq U$, so $\mu(X) \subseteq \mu(U)$.

(4.1)

Trivial.

(6.1)

Follows from (4.1) and (5.2).

(6.2)

Let the prerequisites of $(\mu Cumt\alpha)$ hold.

We first show by induction $\mu(X_\alpha \cup U) \subseteq \mu(U)$.

Proof:

$\alpha = 0 : \mu(X_0) \subseteq U \rightarrow \mu(X_0 \cup U) = \mu(U)$ by (5.1.1). Let for all $\beta < \alpha$ $\mu(X_\beta \cup U) \subseteq \mu(U) \subseteq U$. By prerequisite, $\mu(X_\alpha) \subseteq U \cup \bigcup\{X_\beta : \beta < \alpha\}$, thus $\mu(X_\alpha \cup U) \subseteq \mu(X_\alpha) \cup \mu(U)$ $\subseteq \bigcup\{U \cup X_\beta : \beta < \alpha\}$,

so $\forall \beta < \alpha$ $\mu(X_\alpha \cup U) \cap (U \cup X_\beta) \subseteq \mu(U)$ by (5.1.3), thus $\mu(X_\alpha \cup U) \subseteq \mu(U)$.

Consequently, under the above prerequisites, we have $\mu(X_\alpha \cup U) \subseteq \mu(U) \subseteq U \subseteq U \cup X_\alpha$, so by (μCum) $\mu(U) = \mu(X_\alpha \cup U)$, and, finally, $\mu(U) \cap X_\alpha = \mu(X_\alpha \cup U) \cap X_\alpha \subseteq \mu(X_\alpha)$ by (μPR).

Note that finite unions take us over the limit step, essentially, as all steps collaps, and $\mu(X_\alpha \cup U)$ will always be $\mu(U)$, so there are no real changes.

(5.3)

Follows from (6.2) and (4.1).

(3.1)

Follows from (5.2) and (5.3).

(3.2)

Follows from (6.1) and (6.2).

(4.2)

Follows from (2.3) and (2.1).

(4.3)

Follows from (5.2) and (6.2).

(5.4)

$\mu(X) \subseteq U \rightarrow \mu(X) \subseteq U \cap X \subseteq X \rightarrow \mu(X \cap U) = \mu(X) \rightarrow X \cap \mu(U) = (X \cap U) \cap \mu(U) \subseteq \mu(X \cap U) = \mu(X)$

(3.3)

See Example 2.1.

(6.3)

This is a consequence of (3.3).

(7)

Trivial. Let $X \subseteq Y$, so by $(\mu \subseteq)$ $\mu(X) \subseteq X \subseteq Y$, so by $(\mu Cum0)$ $X \cap \mu(Y) \subseteq \mu(X)$.

(8.1)

Trivial by definition of $H(U,x)$.

(8.2)

Let $x \in \mu(U)$, $x \in Y$, $\mu(Y) \subseteq H(U,x)$ (and thus $Y \subseteq H(U,x)$ by definition). Thus, we have a sequence $X_0 := U$, $\mu(X_\beta) \subseteq U \cup \bigcup\{X_\gamma : \gamma < \beta\}$ and $Y = X_\alpha$ for some α (after X_0, enumerate arbitrarily $H(U,x)_1$, then $H(U,x)_2$, etc., do nothing at limits). So $x \in \bigcap\{X_\gamma : \gamma \leq \alpha\} \cap \mu(U)$, and $x \in \mu(X_\alpha) = \mu(Y)$ by $(\mu Cum\infty)$. Remark: The same argument shows that we can replace "$x \in X$" equivalently by "$x \in \mu(X)$" in the definition of $H(U,x)_{\alpha+1}$, as was done in Definition 3.7.5 in [Sch04].

(8.3)

Suppose $(\mu Cum\alpha)$ fails, we show that then so does (HUx). As $(\mu Cum\alpha)$ fails, for all $\beta \leq \alpha$ $\mu(X_\beta) \subseteq U \cup \bigcup\{X_\gamma : \gamma < \beta\}$, but there is $x \in \bigcap\{X_\gamma : \gamma \leq \alpha\} \cap \mu(U)$, $x \notin \mu(X_\alpha)$. Thus for all $\beta \leq \alpha$ $\mu(X_\beta) \subseteq X_\beta \subseteq H(U,x)$, moreover $x \in \mu(U)$, $x \in X_\alpha - \mu(X_\alpha)$, but $\mu(X_\alpha) \subseteq H(U,x)$, so (HUx) fails.

□

We turn to $H(U)$.

Fact 2.2

Let A, X, U, U', Y and all A_i be in $\mathcal{Y}$.

(1) $(\mu \subseteq)$ and (HU) entail:

(1.1) (μPR)

(1.2) (μCum)

(2) (HU) + $(\cup)$ $\rightarrow$ (HUx)

(3) $(\mu \subseteq)$ and (μPR) entail:

(3.1) $A = \bigcup\{A_i : i \in I\} \rightarrow \mu(A) \subseteq \bigcup\{\mu(A_i) : i \in I\}$,

(3.2) $U \subseteq H(U)$, and $U \subseteq U' \rightarrow H(U) \subseteq H(U')$,

(3.3) $\mu(U \cup Y) - H(U) \subseteq \mu(Y)$ - if $\mu(U \cup Y)$ is defined, in particular, if $(\cup)$ holds.

(4) $(\cup)$, $(\mu \subseteq)$, (μPR), (μCUM) entail:

(4.1) $H(U) = H_1(U)$

(4.2) $U \subseteq A$, $\mu(A) \subseteq H(U) \rightarrow \mu(A) \subseteq U$, (4.3) $\mu(Y) \subseteq H(U) \to Y \subseteq H(U)$ and $\mu(U \cup Y) = \mu(U)$,

(4.4) $x \in \mu(U)$, $x \in Y - \mu(Y) \to Y \not\subseteq H(U)$ (and thus (HU)),

(4.5) $Y \not\subseteq H(U) \to \mu(U \cup Y) \not\subseteq H(U)$.

(5) $(\cup)$, $(\mu \subseteq)$, (HU) entail

(5.1) $H(U) = H_1(U)$

(5.2) $U \subseteq A$, $\mu(A) \subseteq H(U) \to \mu(A) \subseteq U$,

(5.3) $\mu(Y) \subseteq H(U) \to Y \subseteq H(U)$ and $\mu(U \cup Y) = \mu(U)$,

(5.4) $x \in \mu(U)$, $x \in Y - \mu(Y) \to Y \not\subseteq H(U)$,

(5.5) $Y \not\subseteq H(U) \to \mu(U \cup Y) \not\subseteq H(U)$.

Proof:

(1.1) By (HU), if $\mu(Y) \subseteq H(U)$, then $\mu(U) \cap Y \subseteq \mu(Y)$. But, if $Y \subseteq U$, then $\mu(Y) \subseteq H(U)$ by $(\mu \subseteq)$.

(1.2) Let $\mu(U) \subseteq X \subseteq U$. Then by (1.1) $\mu(U) = \mu(U) \cap X \subseteq \mu(X)$. By prerequisite, $\mu(U) \subseteq U \subseteq H(X)$, so $\mu(X) = \mu(X) \cap U \subseteq \mu(U)$ by $(\mu \subseteq)$.

(2) By (1.2), (HU) entails (μCum), so by $(\cup)$ and Fact 2.1, (5.2) $(\mu Cum\infty)$ holds, so by Fact 2.1, (8.2) (HUx) holds.

(3.1) $\mu(A) \cap A_j \subseteq \mu(A_j) \subseteq \bigcup \mu(A_i)$, so by $\mu(A) \subseteq A = \bigcup A_i$ $\mu(A) \subseteq \bigcup \mu(A_i)$.

(3.2) trivial.

(3.3) $\mu(U \cup Y) - H(U) \subseteq_{(3.2)} \mu(U \cup Y) - U \subseteq$ (by $(\mu \subseteq)$ and (3.1)) $\mu(U \cup Y) \cap Y \subseteq_{(\mu PR)} \mu(Y)$.

(4.1) We show that, if $X \subseteq H_2(U)$, then $X \subseteq H_1(U)$, more precisely, if $\mu(X) \subseteq H_1(U)$, then already $X \subseteq H_1(U)$, so the construction stops already at $H_1(U)$. Suppose then $\mu(X) \subseteq \bigcup\{Y : \mu(Y) \subseteq U\}$, and let $A := X \cup U$. We show that $\mu(A) \subseteq U$, so $X \subseteq A \subseteq H_1(U)$. Let $a \in \mu(A)$. By (3.1), $\mu(A) \subseteq \mu(X) \cup \mu(U)$. If $a \in \mu(U) \subseteq U$, we are done. If $a \in \mu(X)$, there is Y s.t. $\mu(Y) \subseteq U$ and $a \in Y$, so $a \in \mu(A) \cap Y$. By Fact 2.1, (5.1.3), we have for Y s.t. $\mu(Y) \subseteq U$ and $U \subseteq A$ $\mu(A) \cap Y \subseteq \mu(U)$. Thus $a \in \mu(U)$, and we are done again.

(4.2) Let $U \subseteq A$, $\mu(A) \subseteq H(U) = H_1(U)$ by (4.1). So $\mu(A) = \bigcup\{\mu(A) \cap Y : \mu(Y) \subseteq U\}$ $\subseteq \mu(U) \subseteq U$, again by Fact 2.1, (5.1.3).

(4.3) Let $\mu(Y) \subseteq H(U)$, then by $\mu(U) \subseteq H(U)$ and (3.1) $\mu(U \cup Y) \subseteq \mu(U) \cup \mu(Y) \subseteq H(U)$, so by (4.2) $\mu(U \cup Y) \subseteq U$ and $U \cup Y \subseteq H(U)$. Moreover, $\mu(U \cup Y) \subseteq U \subseteq U \cup Y \to_{(\mu CUM)}$ $\mu(U \cup Y) = \mu(U)$.

(4.4) If not, $Y \subseteq H(U)$, so $\mu(Y) \subseteq H(U)$, so $\mu(U \cup Y) = \mu(U)$ by (4.3), but $x \in Y - \mu(Y)$

$\rightarrow_{(\mu PR)}$ $x \notin \mu(U \cup Y) = \mu(U)$, *contradiction.*

(4.5) $\mu(U \cup Y) \subseteq H(U) \rightarrow_{(4.3)} U \cup Y \subseteq H(U)$.

(5) Trivial by (1) and (4).

□

(5) is just noted for the convenience of the reader. It will be used for the proof of Fact 2.3.

Thus, in the presence of $(\cup)$ $H(U,x)$ can be simplified to $H(U)$, which is constructed in one single step, and is independent from x. Of course, then $H(U,x) \subseteq H(U)$.

2.3 Representation by smooth preferential structures

2.3.1 The not necessarily transitive case

We adapt Proposition 3.7.15 in [Sch04] and its proof. All we need is (HUx) and $(\mu \subseteq)$. We modify the proof of Remark 3.7.13 (1) in [Sch04] (now Remark 2.4) so we will not need $(\cap)$ any more. We will give the full proof, although its essential elements have already been published, for three reasons: First, the new version will need less prerequisites than the old proof does (closure under finite intersections is not needed any more, and replaced by (HUx)). Second, we will more clearly separate the requirements to do the construction from the construction itself, thus splitting the proof neatly into two parts.

We show how to work with $(\mu \subseteq)$ and (HUx) only. Thus, once we have shown $(\mu \subseteq)$ and (HUx), we have finished the substantial side, and enter the administrative part, which will not use any prerequisites about domain closure any more. At the same time, this gives a uniform proof of the difficult part for the case with and without $(\cup)$, in the former case we can even work with the stronger $H(U)$. The easy direction of the former parts needs a proof of the stronger $H(U)$, but this is easy.

Note that, by Fact 2.1, (8.3) and (7), (HUx) entails (μPR), so we can use it in our context, where (HUx) will be the central property.

Fact 2.3

(1) $x \in \mu(Y)$, $\mu(Y) \subseteq H(U,x) \rightarrow Y \subseteq H(U,x)$,

(2) (HUx) holds in all smooth models.

Proof:

(1) Trivial by definition.

(2) Suppose not. So let $x \in \mu(U)$, $x \in Y - \mu(Y)$, $\mu(Y) \subseteq H(U,x)$. By smoothness, there is $x_1 \in \mu(Y)$, $x \succ x_1$, and let κ_1 be the least κ s.t. $x_1 \in H(U,x)_{\kappa_1}$. κ_1 is not a limit, and $x_1 \in U'_{x_1} - \mu(U'_{x_1})$ with $x \in \mu(U'_{x_1})$ for some U'_{x_1}, so as $x_1 \notin \mu(U'_{x_1})$, there must be (by smoothness) some other $x_2 \in \mu(U'_{x_1}) \subseteq H(U,x)_{\kappa_1 - 1}$ with $x \succ x_2$. Continue with x_2, we thus construct a descending chain of ordinals, which cannot be infinite, so there must be $x_n \in \mu(U'_{x_n}) \subseteq U$, $x \succ x_n$, contradicting minimality of x in U. (More precisely, this works for all copies of x.) $\square$

We first show two basic facts and then turn to the main result, Proposition 2.6.

Definition 2.3

For $x \in Z$, let $\mathcal{W}_x := \{\mu(Y) \colon Y \in \mathcal{Y} \wedge x \in Y - \mu(Y)\}$, $\Gamma_x := \Pi \mathcal{W}_x$, and $K := \{x \in Z \colon \exists X \in \mathcal{Y}.x \in \mu(X)\}$.

Remark 2.4

(1) $x \in K \rightarrow \Gamma_x \neq \emptyset$,

(2) $g \in \Gamma_x \rightarrow ran(g) \subseteq K$.

Proof:

(1) We give two proofs, the first uses $(\mu Cum0)$, the second the stronger (HUx).

(a) We have to show that $Y \in \mathcal{Y}$, $x \in Y - \mu(Y) \rightarrow \mu(Y) \neq \emptyset$. Suppose then $x \in \mu(X)$, this exists, as $x \in K$, and $\mu(Y) = \emptyset$, so $\mu(Y) \subseteq X$, $x \in Y$, so by $(\mu Cum0)$ $x \in \mu(Y)$.

(b) $\mu(Y) = \emptyset \rightarrow Y \subseteq H(U,x)$, contradicting $x \in Y - \mu(Y)$.

(2) By definition, $\mu(Y) \subseteq K$ for all $Y \in \mathcal{Y}$. $\square$

Claim 2.5

Let $U \in \mathcal{Y}$, $x \in K$. Then

(1) $x \in \mu(U) \leftrightarrow x \in U \wedge \exists f \in \Gamma_x.ran(f) \cap U = \emptyset$,

(2) $x \in \mu(U) \leftrightarrow x \in U \wedge \exists f \in \Gamma_x.ran(f) \cap H(U,x) = \emptyset$.

Proof:

(1)

Case 1: $\mathcal{W}_x = \emptyset$, thus $\Gamma_x = \{\emptyset\}$.

"$\rightarrow$": Take $f := \emptyset$.

"$\leftarrow$": $x \in U \in \mathcal{Y}$, $\mathcal{W}_x = \emptyset \rightarrow x \in \mu(U)$ by definition of $\mathcal{W}_x$.

Case 2: $\mathcal{W}_x \neq \emptyset$.

"$\rightarrow$": Let $x \in \mu(U) \subseteq U$. By (HUx), if $Y \in \mathcal{W}_x$, then $\mu(Y) - H(U,x) \neq \emptyset$.

"$\leftarrow$": If $x \in U - \mu(U)$, $\mu(U) \in \mathcal{W}_x$, moreover $\Gamma_x \neq \emptyset$ by Remark 2.4, (1) and thus $\mu(U) \neq \emptyset$, so $\forall f \in \Gamma_x . ran(f) \cap U \neq \emptyset$.

(2): The proof is verbatim the same as for (1).

$\square$ (Claim 2.5)

The following Proposition 2.6 is the main result of Section 2.3.1 and shows how to characterize smooth structures in the absence of closure under finite unions. The strategy of the proof follows closely the proof of Proposition 3.3.4 in [Sch04].

Proposition 2.6

Let $\mu : \mathcal{Y} \rightarrow \mathcal{P}(Z)$. Then there is a $\mathcal{Y}-$smooth preferential structure $\mathcal{Z}$, s.t. for all $X \in \mathcal{Y}$ $\mu(X) = \mu_{\mathcal{Z}}(X)$ iff μ satisfies $(\mu \subseteq)$ and (HUx) above.

Proof:

"$\rightarrow$" (HUx) was shown in Fact 2.3.

Outline of "$\leftarrow$": We first define a structure $\mathcal{Z}$ which represents μ, but is not necessarily $\mathcal{Y}-$smooth, refine it to $\mathcal{Z}'$ and show that $\mathcal{Z}'$ represents μ too, and that $\mathcal{Z}'$ is $\mathcal{Y}-$smooth.

In the structure $\mathcal{Z}'$, all pairs destroying smoothness in $\mathcal{Z}$ are successively repaired, by adding minimal elements: If $< y, j >$ is not minimal, and has no minimal $< x, i >$ below it, we just add one such $< x, i >$. As the repair process might itself generate such "bad" pairs, the process may have to be repeated infinitely often. Of course, one has to take care that the representation property is preserved.

Construction 2.1

(Construction of $\mathcal{Z}$)

Let $\mathcal{X} := \{< x, g >: x \in K, g \in \Gamma_x\}$, $< x', g' > \prec < x, g > :\leftrightarrow x' \in ran(g)$, $\mathcal{Z} := < \mathcal{X}, \prec >$.

Claim 2.7

$\forall U \in \mathcal{Y}.\mu(U) = \mu_{\mathcal{Z}}(U)$

Proof:

Case 1: $x \notin K$. Then $x \notin \mu(U)$ and $x \notin \mu_{\mathcal{Z}}(U)$.

Case 2: $x \in K$.

By Claim 2.5, (1) it suffices to show that for all $U \in \mathcal{Y}$ $x \in \mu_{\mathcal{Z}}(U) \leftrightarrow x \in U \wedge \exists f \in \Gamma_x.ran(f) \cap U = \emptyset$. Fix $U \in \mathcal{Y}$.

"$\rightarrow$": $x \in \mu_{\mathcal{Z}}(U) \rightarrow$ ex. $< x, f >$ minimal in $\mathcal{X}\lceil U$, thus $x \in U$ and there is no $< x', f' > \prec < x, f >$, $x' \in U$, $x' \in K$. But if $x' \in K$, then by Remark 2.4, (1), $\Gamma_{x'} \neq \emptyset$, so we find suitable f'. Thus, $\forall x' \in ran(f).x' \notin U$ or $x' \notin K$. But $ran(f) \subseteq K$, so $ran(f) \cap U = \emptyset$.

"$\leftarrow$": If $x \in U$, $f \in \Gamma_x$ s.t. $ran(f) \cap U = \emptyset$, then $< x, f >$ is minimal in $\mathcal{X}\lceil U$. $\Box$ (Claim 2.7)

We now construct the refined structure $\mathcal{Z}'$.

Construction 2.2

(Construction of $\mathcal{Z}'$)

σ is called x-admissible sequence iff

1. σ is a sequence of length $\leq \omega$, $\sigma = \{\sigma_i : i \in \omega\}$,
2. $\sigma_o \in \Pi\{\mu(Y)$: $Y \in \mathcal{Y} \wedge x \in Y - \mu(Y)\}$,
3. $\sigma_{i+1} \in \Pi\{\mu(X)$: $X \in \mathcal{Y} \wedge x \in \mu(X) \wedge ran(\sigma_i) \cap X \neq \emptyset\}$.

By 2., σ_0 minimizes x, and by 3., if $x \in \mu(X)$, and $ran(\sigma_i) \cap X \neq \emptyset$, i.e. we have destroyed minimality of x in X, x will be above some y minimal in X to preserve smoothness.

Let Σ_x be the set of x-admissible sequences, for $\sigma \in \Sigma_x$ let $\widehat{\sigma} := \bigcup\{ran(\sigma_i) : i \in \omega\}$. Note that by Remark 2.4, (1), $\Sigma_x \neq \emptyset$, if $x \in K$ (this does σ_0, σ_{i+1} is trivial as by prerequisite $\mu(X) \neq \emptyset$).

Let $\mathcal{X}' := \{< x, \sigma >$: $x \in K \wedge \sigma \in \Sigma_x\}$ and $< x', \sigma' > \prec' < x, \sigma > :\leftrightarrow x' \in \widehat{\sigma}$. Finally, let $\mathcal{Z}' :=< \mathcal{X}', \prec' >$, and $\mu' := \mu_{\mathcal{Z}'}$.

It is now easy to show that $\mathcal{Z}'$ represents μ, and that $\mathcal{Z}'$ is smooth. For $x \in \mu(U)$, we construct a special x-admissible sequence $\sigma^{x,U}$ using the properties of $H(U, x)$ as described at the beginning of this section.

Claim 2.8

For all $U \in \mathcal{Y}$ $\mu(U) = \mu_{\mathcal{Z}}(U) = \mu'(U)$.

Proof:

If $x \notin K$, then $x \notin \mu_{\mathcal{Z}}(U)$, and $x \notin \mu'(U)$ for any U. So assume $x \in K$. If $x \in U$ and $x \notin \mu_{\mathcal{Z}}(U)$, then for all $< x, f > \in \mathcal{X}$, there is $< x', f' > \in \mathcal{X}$ with $< x', f' > \prec < x, f >$ and $x' \in U$. Let now $< x, \sigma > \in \mathcal{X}'$, then $< x, \sigma_0 > \in \mathcal{X}$, and let $< x', f' > \prec < x, \sigma_0 >$ in $\mathcal{Z}$ with $x' \in U$. As $x' \in K$, $\Sigma_{x'} \neq \emptyset$, let $\sigma' \in \Sigma_{x'}$. Then $< x', \sigma' > \prec' < x, \sigma >$ in $\mathcal{Z}'$. Thus $x \notin \mu'(U)$. Thus, for all $U \in \mathcal{Y}$, $\mu'(U) \subseteq \mu_{\mathcal{Z}}(U) = \mu(U)$.

It remains to show $x \in \mu(U) \rightarrow x \in \mu'(U)$.

Assume $x \in \mu(U)$ (so $x \in K$), $U \in \mathcal{Y}$, we will construct minimal σ, i.e. show that there is $\sigma^{x,U} \in \Sigma_x$ s.t. $\widehat{\sigma^{x,U}} \cap U = \emptyset$. We construct this $\sigma^{x,U}$ inductively, with the stronger property that $ran(\sigma_i^{x,U}) \cap H(U,x) = \emptyset$ for all $i \in \omega$.

$\underline{\sigma_0^{x,U}:}$

$x \in \mu(U)$, $x \in Y - \mu(Y) \rightarrow \mu(Y) - H(U,x) \neq \emptyset$ by (HUx). Let $\sigma_0^{x,U} \in \Pi\{\mu(Y) - H(U,x) : Y \in \mathcal{Y}, x \in Y - \mu(Y)\}$, so $ran(\sigma_0^{x,U}) \cap H(U,x) = \emptyset$.

$\underline{\sigma_i^{x,U} \rightarrow \sigma_{i+1}^{x,U}:}$

By the induction hypothesis, $ran(\sigma_i^{x,U}) \cap H(U,x) = \emptyset$. Let $X \in \mathcal{Y}$ be s.t. $x \in \mu(X)$, $ran(\sigma_i^{x,U}) \cap X \neq \emptyset$. Thus $X \not\subseteq H(U,x)$, so $\mu(X) - H(U,x) \neq \emptyset$ by Fact 2.3, (1). Let $\sigma_{i+1}^{x,U} \in \Pi\{\mu(X) - H(U,x) : X \in \mathcal{Y}, x \in \mu(X), ran(\sigma_i^{x,U}) \cap X \neq \emptyset\}$, so $ran(\sigma_{i+1}^{x,U}) \cap H(U,x) = \emptyset$. The construction satisfies the x-admissibility condition. □

It remains to show:

Claim 2.9

$\mathcal{Z}'$ is $\mathcal{Y}-$smooth.

Proof:

Let $X \in \mathcal{Y}$, $< x, \sigma > \in \mathcal{X}' \lceil X$.

Case 1, $x \in X - \mu(X)$: Then $ran(\sigma_0) \cap \mu(X) \neq \emptyset$, let $x' \in ran(\sigma_0) \cap \mu(X)$. Moreover, $\mu(X) \subseteq K$. Then for all $< x', \sigma' > \in \mathcal{X}'$ $< x', \sigma' > \prec < x, \sigma >$. But $< x', \sigma^{x',X} >$ as constructed in the proof of Claim 2.8 is minimal in $\mathcal{X}' \lceil X$.

Case 2, $x \in \mu(X) = \mu_{\mathcal{Z}}(X) = \mu'(X)$: If $< x, \sigma >$ is minimal in $\mathcal{X}' \lceil X$, we are done. So suppose there is $< x', \sigma' > \prec < x, \sigma >$, $x' \in X$. Thus $x' \in \widehat{\sigma}$. Let $x' \in ran(\sigma_i)$. So $x \in \mu(X)$ and $ran(\sigma_i) \cap X \neq \emptyset$. But $\sigma_{i+1} \in \Pi\{\mu(X')$: $X' \in \mathcal{Y} \wedge x \in \mu(X') \wedge ran(\sigma_i) \cap X' \neq \emptyset\}$, so X is one of the X', moreover $\mu(X) \subseteq K$, so there is $x'' \in \mu(X) \cap ran(\sigma_{i+1}) \cap K$, so for all $< x'', \sigma'' > \in \mathcal{X}'$ $< x'', \sigma'' > \prec < x, \sigma >$. But again $< x'', \sigma^{x'',X} >$ as constructed in the proof of Claim 2.8 is minimal in $\mathcal{X}' \lceil X$.

$\square$ (Claim 2.9 and Proposition 2.6)

We conclude this section by showing that we cannot improve substantially.

Proposition 2.10

There is no fixed size characterization of $\mu-$functions which are representable by smooth structures, if the domain is not closed under finite unions.

Proof:

Suppose we have a fixed size characterization, which allows to distinguish $\mu-$functions on domains which are not necessarily closed under finite unions, and which can be represented by smooth structures, from those which cannot be represented in this way. Let the characterization have α parameters for sets, and consider Example 2.1 with $\kappa = \beta + 1$, $\beta > \alpha$ (as a cardinal). This structure cannot be represented, as $(\mu Cum\kappa)$ fails. As we have only α parameters, at least one of the X_γ is not mentioned, say X_δ. Wlog, we may assume that $\delta = \delta' + 1$. We change now the structure, and erase one pair of the relation, $x_\delta \prec x_{\delta+1}$. Thus, $\mu(X_\delta) = \{c, x_\delta, x_{\delta+1}\}$. But now we cannot go any more from $X_{\delta'}$ to $X_{\delta'+1} = X_\delta$, as $\mu(X_\delta) \not\subseteq X_{\delta'}$. Consequently, the only chain showing that $(\mu Cum\infty)$ fails is interrupted - and we have added no new possibilities, as inspection of cases shows. ($x_{\delta+1}$ is now globally minimal, and increasing $\mu(X)$ cannot introduce new chains, only interrupt chains.) Thus, $(\mu Cum\infty)$ holds in the modified example, and it is thus representable by a smooth structure, as above proposition shows. As we did not touch any of the parameters, the truth value of the characterization is unchanged, which was negative. So the "characterization" cannot be correct. $\square$

2.3.2 The transitive case

Unfortunately, $(\mu Cumt\infty)$ is a necessary but not sufficient condition for smooth transitive structures, as can be seen in the following example.

Example 2.2

We assume no closure whatever.

$U := \{u_1, u_2, u_3, u_4\}$, $\mu(U) := \{u_3, u_4\}$

$Y_1 := \{u_4, v_1, v_2, v_3, v_4\}$, $\mu(Y_1) := \{v_3, v_4\}$

$Y_{2,1} := \{u_2, v_2, v_4\}$, $\mu(Y_{2,1}) := \{u_2, v_2\}$

$Y_{2,2} := \{u_1, v_1, v_3\}$, $\mu(Y_{2,2}) := \{u_1, v_1\}$

For no A,B $\mu(A) \subseteq B$ $(A \neq B)$, so the prerequisite of $(\mu Cumt\alpha)$ is false, and $(\mu Cumt\alpha)$ holds, but there is no smooth transitive representation possible: if $u_4 \succ v_3$, then $Y_{2,2}$ makes this impossible, if $u_4 \succ v_4$, then $Y_{2,1}$ makes this impossible.

□

Remark 2.11

(1) The situation does not change when we have copies, the same argument will still work: There is a U-minimal copy $< u_4, i >$, by smoothness and Y_1, there must be a Y_1-minimal copy, e.g. $< v_3, j > \prec < u_4, i >$. By smoothness and $Y_{2,2}$, there must be a $Y_{2,2}-$minimal $< u_1, k >$ or $< v_1, l >$ below $< v_3, j >$. But v_1 is in Y_1, contradicting minimality of $< v_3, j >$, u_1 is in U, contadicting minimality of $< u_4, i >$ by transitivity. If we choose $< v_4, j >$ minimal below $< u_4, i >$, we will work with $Y_{2,1}$ instead of $Y_{2,2}$.

(2) We can also close under arbitrary intersections, and the example will still work: We have to consider $U \cap Y_1$, $U \cap Y_{2,1}$, $U \cap Y_{2,2}$, $Y_{2,1} \cap Y_{2,2}$, $Y_1 \cap Y_{2,1}$, $Y_1 \cap Y_{2,2}$, there are no further intersections to consider. We may assume $\mu(A) = A$ for all these intersections (working with copies). But then $\mu(A) \subseteq B$ implies $\mu(A) = A$ for all sets, and all $(\mu Cumt\alpha)$ hold again trivially.

(3) If we had finite unions, we could form $A := U \cup Y_1 \cup Y_{2,1} \cup Y_{2,2}$, then $\mu(A)$ would have to be a subset of $\{u_3\}$ by (μPR), so by (μCUM) $u_4 \notin \mu(U)$, a contradiction. Finite unions allow us to "look ahead", without $(\cup)$, we see desaster only at the end - and have to backtrack, i.e. try in our example $Y_{2,1}$, once we have seen impossibility via $Y_{2,2}$, and discover impossibility again at the end.

We discuss and define now an analogon to (HU) or (HUx), condition $(\mu\tau)$, defined in Definition 2.4.

The different possible cases The problem is to minimize an element, already minimal in a finite number of sets, in a new set, without destroying previous minimality. We have to examine the way the new minimal element is chosen.

(I) Going forward

Let $Y_0, \ldots, Y_n$ be treated and $x_n \in \mu(Y_n)$. (We can argue without copies, as we may assume that we have chosen a minimal copy.)

Treating Y_{n+1} :

For all Y_{n+1} s.t. $x_n \in Y_{n+1}$, we have to treat Y_{n+1} and choose $x_{n+1} \in \mu(Y_{n+1})$.

If $x_n \notin Y_{n+1}$, there is nothing to do: Y_{n+1} is not to be considered.

Case 1: $x_n \in \mu(Y_{n+1})$, then we can either

Case 1.1: leave it as it is, i.e. $x_{n+1} := x_n$,

Case 1.2: minimize it by another $x_{n+1} \in \mu(Y_{n+1})$, outside $Y_0 \cup \ldots \cup Y_n$ as we do not want to destroy previous minimality - assuming that $\mu(Y_{n+1}) \not\subseteq Y_0 \cup \ldots \cup Y_n$.

Case 2: $x_n \in Y_{n+1} - \mu(Y_{n+1})$, then we have to

minimize it by another $x_{n+1} \in \mu(Y_{n+1})$, outside $Y_0 \cup \ldots \cup Y_n$ as we do not want to destroy previous minimality - assuming that $\mu(Y_{n+1}) \not\subseteq Y_0 \cup \ldots \cup Y_n$.

We tentatively write this down as follows:

- $(Y_{n+1}, x_{n+1} \in \mu(Y_{n+1}), m)$ - if we modified it, i.e. $x_{n+1} \neq x_n$, m for modify, and

- $(Y_{n+1}, x_{n+1} \in \mu(Y_{n+1}), c)$ - if $x_{n+1} = x_n$, c for constant.

We have to do this for all Y_{n+1} s.t. $x_n \in Y_{n+1}$.

We continue to go forward one further step.

Treating Y_{n+2} :

For all Y_{n+2} s.t. $x_{n+1} \in Y_{n+2}$, we treat Y_{n+2} by choosing $x_{n+2} \in \mu(Y_{n+2})$.

Let $x_{n+1} \in Y_{n+2}$.

Case 1: $x_{n+1} \in \mu(Y_{n+2})$

Case 1.1: We leave x_{n+1} as it is, $x_{n+2} := x_{n+1}$, so the next element is $(Y_{n+2}, x_{n+2} \in \mu(Y_{n+2}), c)$.

Case 1.2: We try to modify.

If $\mu(Y_{n+2}) \not\subseteq Y_0 \cup \ldots \cup Y_n \cup Y_{n+1}$, then we can choose any $x_{n+2} \in \mu(Y_{n+2}) - Y_0 \cup \ldots \cup Y_n \cup Y_{n+1}$, and the successor is $(Y_{n+2}, x_{n+2} \in \mu(Y_{n+2}), m)$.

If $\mu(Y_{n+2}) \subseteq Y_0 \cup \ldots \cup Y_n \cup Y_{n+1}$, then this is impossible, and we have to work with Case 1.1.

Case 2: $x_{n+1} \in Y_{n+2} - \mu(Y_{n+2})$

We have to modify.

If $\mu(Y_{n+2}) \not\subseteq Y_0 \cup \ldots \cup Y_n \cup Y_{n+1}$, then we can choose any $x_{n+2} \in \mu(Y_{n+2}) - Y_0 \cup \ldots \cup Y_n \cup Y_{n+1}$, and the successor is $(Y_{n+2}, x_{n+2} \in \mu(Y_{n+2}), m)$

If $\mu(Y_{n+2}) \subseteq Y_0 \cup \ldots \cup Y_n \cup Y_{n+1}$, then this is impossible, but now we have no alternative, thus already the choice of $x_{n+1} \in Y_{n+2} - \mu(Y_{n+2})$ was impossible, as we then have to consider Y_{n+2}.

This is the heart of the problem: a subsequent step (considering Y_{n+2}) may show that a previous choice (x_{n+1}) was impossible, so we have to backtrack.

Of course, this does not only eliminate this particular x_{n+1}, but any other $x_{n+1} \in Y_{n+2} - \mu(Y_{n+2})$, too. But it does not concern any $x_{n+1} \in \mu(Y_{n+2})$, as we then have alternative 1.1 above. We also note that it is unimportant how we obtained the previous x_{n+1} - by modification or staying constant.

(II) Backtracking

We discuss now the repercussions of such, in hindsight, impossible choices.

Treating Y_{n+1} :

Let $\mu(Y_{n+2}) \subseteq Y_0 \cup \ldots \cup Y_n \cup Y_{n+1}$ and $x_n \in Y_{n+1}$, so we have to treat Y_{n+1}, and choose $x_{n+1} \in \mu(Y_{n+1})$. If we choose $x_{n+1} \in Y_{n+2} - \mu(Y_{n+2})$, then the next step will show us that this is impossible, so we have to choose x_{n+1} outside of $Y_{n+2} - \mu(Y_{n+2})$. If $x_n \in \mu(Y_{n+1})$, and $x_n \notin Y_{n+2} - \mu(Y_{n+2})$, we can choose $x_{n+1} := x_n$. If $x_n \notin \mu(Y_{n+1})$, we have to choose $x_{n+1} \in \mu(Y_{n+1}) - (Y_0 \cup \ldots \cup Y_n) - (Y_{n+2} - \mu(Y_{n+2}))$.

So there is an additional problem here: if we can choose $x_{n+1} := x_n$, we have more liberty, we need not necessarily avoid $Y_0 \cup \ldots \cup Y_n$ (if we were constant all the time, we need not avoid any of the previous Y_i). For this reason, the following simplification will not work: Suppose there is a cover of $\mu(Y_{n+1}) - (Y_0 \cup \ldots \cup Y_n)$ by such $Y_{n+2,i} - \mu(Y_{n+2,i})$ with $\mu(Y_{n+2,i}) \subseteq Y_0 \cup \ldots \cup Y_n \cup Y_{n+1}$, then we cannot choose $x_n \in \mu(Y_n) \cap Y_{n+1}$. This will only be true if we cannot choose $x_{n+1} := x_n$. If $x_{n+1} = x_n$ is possible, we can avoid the cover, and simply stay in Y_n, and if $x_{n+1} = x_n = \ldots = x_i$ is possible, we will not need to avoid $Y_i \cup \ldots \cup Y_n$. So the situation seems quite complicated, and we will not win much by considering sets of points, and will therefore consider in the final approach single points.

A more formal treatment We will consider tripels of the form

$< (Y_0, \ldots, Y_n), (x_0, \ldots, x_n), a >$, where

- the sequences are finite,

- a is - or $*$,

- $x_i \in \mu(Y_i)$,

- $x_i \in Y_{i+1}$.

To abbreviate, we also write $< \Sigma, \sigma, a >$.

Just writing $< Y_i, x_i, a >$ would be simpler, we chose above notation for better readability.

$x_0 \in \mu(Y_0)$ is arbitrary.

$< \Sigma, \sigma, * >$ has no successors, it is a dead end.

Let $< (Y_0, \ldots, Y_n), (x_0, \ldots, x_n), - >$ be given. We consider all Y_{n+1} s.t. $Y_{n+1} \notin \{Y_0, \ldots, Y_n\}$ and $x_n \in Y_{n+1}$. (Note: the sequence $x_0, \ldots, x_n$ may be constant, so x_n may be an element of all $\mu(Y_i)$, $0 \leq i \leq n$.)

If there are no such Y_{n+1}, then we are done with this sequence, and $< (Y_0, \ldots, Y_n), (x_0, \ldots, x_n), - >$ has no successors, but it is NOT a dead end - there is simply nothing to treat any more.

Case 1: $x_n \in \mu(Y_{n+1})$.

Case 1.1: $< (Y_0, \ldots, Y_n, Y_{n+1}), (x_0, \ldots, x_n, x_{n+1} := x_n), - >$ is a successor.

Case 1.2: If $\mu(Y_{n+1}) - (Y_0 \cup \ldots \cup Y_n) \neq \emptyset$, then for all $x_{n+1} \in \mu(Y_{n+1}) - (Y_0 \cup \ldots \cup Y_n) \neq \emptyset$ $< (Y_0, \ldots, Y_n, Y_{n+1}), (x_0, \ldots, x_n, x_{n+1}), - >$ is a successor. If If $\mu(Y_{n+1}) - (Y_0 \cup \ldots \cup Y_n) = \emptyset$, then there are no successors of type 1.1.

Case 1: $x_n \in Y_{n+1} - \mu(Y_{n+1})$.

If $\mu(Y_{n+1}) - (Y_0 \cup \ldots \cup Y_n) \neq \emptyset$, then for all $x_{n+1} \in \mu(Y_{n+1}) - (Y_0 \cup \ldots \cup Y_n) \neq \emptyset$ $< (Y_0, \ldots, Y_n, Y_{n+1}), (x_0, \ldots, x_n, x_{n+1}), - >$ is a successor. If If $\mu(Y_{n+1}) - (Y_0 \cup \ldots \cup Y_n) = \emptyset$, then there are no successors of $< (Y_0, \ldots, Y_n), (x_0, \ldots, x_n), - >$, and we mark it is a dead end, by changing the label: $< (Y_0, \ldots, Y_n), (x_0, \ldots, x_n), * >$, as, by $x_n \in Y_{n+1}$, Y_{n+1} has to be treated, but it would lead us back to $Y_0 \cup \ldots \cup Y_n$, destroying previous minimality.

Pruning:

We pass now the labels $*$ downward, against above inductive construction, when needed.

Note that, originally, a node can only have label $*$ if we have to choose $x_{n+1} \neq x_n$. This need not be the case any more now, when we pass $*$ downward.

Consider $< (Y_0, \ldots, Y_n), (x_0, \ldots, x_n), - >$, and suppose for at least one (fixed) Y_{n+1} with $x_n \in Y_{n+1}$ all $x_{n+1} \in \mu(Y_{n+1})$ are already marked $*$, then we change the label of $< (Y_0, \ldots, Y_n), (x_0, \ldots, x_n), - >$, i.e. it will now be $< (Y_0, \ldots, Y_n), (x_0, \ldots, x_n), * >$, i.e. it is a dead end, too. The reason: We have to treat Y_{n+1}, but we cannot, so we must avoid Y_{n+1}, thus x_n cannot be chosen, as $x_n \in Y_{n+1}$.

Finally, if we have to pass $*$ down to the root for some $< (Y_0), (x_0), - >$ the construction fails, and there is no transitive smooth representation.

It is easy to see that the condition is necessary: Take $x_0 \in \mu(Y_0)$. If $x_0 \in Y_1 - \mu(Y_1)$, we have to find $x_1 \in \mu(Y_1)$ below it. It cannot be in Y_0, so choose outside. If $x_1 \in Y_2 - \mu(Y_2)$, we have to minimze it by smoothness by some $x_2 \in \mu(Y_2)$, it cannot be in Y_1 (this would again destroy minimality of x_1), but it cannot be in Y_0 either, as this would, by transitivity, destroy minimality of x_0, etc. Thus we can find a tree as above, where each element and candidate set is treated.

We turn to completeness, but this is almost routine now, as we can do the standard administrative part.

Before we do so, we will, however, give the condition a name, and give a simple example and result:

Definition 2.4

$(\mu\tau)$ is the property, that for each $U \in \mathcal{Y}$ and $x \in \mu(U)$ above construction can be carried out, i.e. $*$ will not be propagated down to the root.

We first present some results for $(\mu\tau)$, before giving a completeness proof.

The following example illustrates the situations for $(\mu\tau)$ and (HU).

Example 2.3

Let $x \in \mu(U)$.

Let $x > y_1 > y_2 > y_3 > \ldots.$ and $x > z_1$, $y_1 > z_2$, $y_2 > z_3$, $\ldots$, and let there be chains of length n to come back from z_n to U, e.g. $z_2 > u_1 > u_2$ with $u_2 \in U$.

Let $Y_1 := \{x, y_1, z_1\}$, $Y_2 := \{y_1, y_2, z_2\}$, $Y_3 := \{y_2, y_3, z_3\}$, $U_{2,1} := \{z_2, u_1\}$, $U_{2,2} := \{u_1, u_2\}$ etc.

Then there is a branch $x > y_1 > y_2 > y_3 > \ldots.$ which we can choose, which will never come back to U, and none of the Y_i is a subset of $H(U)$.

Fact 2.12

(1) $(\mu\tau) \to (HU)$

(2) $(\mu\tau) \to (\mu Cumt\infty)$

(3) $(\cup) + (HU) + (\mu \subseteq) \to (\mu\tau)$

Proof:

(1) Let $(\mu\tau)$ hold and (HU) fail, so there is Y with $x \in \mu(U)$, $x \in Y - \mu(Y)$, $Y \subseteq H(U)$. By $(\mu\tau)$, there is a tree beginning at x, and choosing some $y \in \mu(Y)$. Let α_y be the first α s.t. $y \in H_\alpha(U)$. So for some Y_1 $y \in Y_1 - \mu(Y_1)$ and $\mu(Y_1) \subseteq H_{\alpha_y - 1}(U)$. By construction, the tree must choose some $y_1 \in \mu(Y_1)$. Let α_{y_1} be the first α s.t. $y_1 \in H_\alpha(U)$, so for some Y_2 $y_1 \in Y_2 - \mu(Y_2)$, and $\mu(Y_2) \subseteq H_{\alpha_{y_1}-1}(U)$. Again, we must choose some $y_2 \in \mu(Y_2)$, resulting in α_{y_2}, etc. This results in a descending chain of ordinals, $\alpha_y > \alpha_{y_1} > \alpha_{y_2} >$ etc., which cannot be infinite, so it has to end in U, a contradiction, as we are not allowed to come back to U. Consequently, the tree cannot go from x to $y \in H(U)$, so by construction $\mu(Y) \not\subseteq H(U)$.

(2) The proof is almost verbatim the same as for (1).

Let $(\mu\tau)$ hold, and $(\mu Cumt\gamma)$ fail for some γ, so there is X_α, $x \in \mu(U)$, $x \in X_\alpha - \mu(X_\alpha)$, and for all $\beta \leq \alpha$ $\mu(X_\beta) \subseteq U \cup \bigcup\{X_\gamma : \gamma < \beta\}$. By $(\mu\tau)$, there is a tree beginning at x, choosing $y \in \mu(X_\alpha)$. Let α_y be the first β s.t. $y \in U \cup \bigcup\{X_\gamma : \gamma < \beta\}$. So for some Y_1 $y \in Y_1 - \mu(Y_1)$, and $\mu(Y_1) \subseteq U \cup \bigcup\{X_\gamma : \gamma < \beta' < \beta\}$. By construction, the tree must choose some $y_1 \in \mu(Y_1)$. Let α_{y_1} be the first β s.t. $y_1 \in U \cup \bigcup\{X_\gamma : \gamma < \beta\}$ etc. Again, we have a finite sequence going back to U, so we cannot choose in such X_α, and x cannot be in $\mu(U) \cap (X_\alpha - \mu(X_\alpha))$.

(3) We use Fact 2.2 repeatedly, the references are for this fact.

We construct a tree using the same idea as e.g. in the proof of Claim 3.3.6 in [Sch04]. Let $x \in \mu(U)$, $x \in Y - \mu(Y)$, then by (5.4), $Y \not\subseteq H(U)$, so we can choose $y \in \mu(U \cup Y) - H(U) \subseteq \mu(Y)$ by (5.5) and (3.3). Let now $y \in Y_1 - \mu(Y_1)$. As $y \in \mu(U \cup Y)$, $Y_1 \not\subseteq H(U \cup Y)$ by (5.4), so we can choose $y_2 \in \mu(U \cup Y \cup Y_1) - H(U \cup Y) \subseteq \mu(Y_1)$ again by (5.5) and (3.3), etc. Thus, coming back to any earlier set is avoided, and we can build a tree as wanted.

□

The administrative part of the proof The proof is now almost finished, as a matter of fact, we can take (the final part of) the proof of Proposition 3.3.8, Construction 3.3.3 in [Sch04].

Proposition 2.13

If $\mu : \mathcal{Y} \to \mathcal{P}(Z)$, then there is a $\mathcal{Y}-$smooth transitive preferential structure $\mathcal{Z}$, s.t. for all $X \in \mathcal{Y}$ $\mu(X) = \mu_{\mathcal{Z}}(X)$ iff μ satisfies $(\mu \subseteq)$, (μPR), (μCum), $(\mu\tau)$.

Proof:

(A) The easy direction:

(B) The harder direction

We will suppose for simplicity that $Z = K$ - the general case in easy to obtain by a technique similar to that in Section 3.3.1 of [Sch04], but complicates the picture.

The relation $\prec$ between trees will essentially be determined by the subtree relation.

Definition 2.5

If t is a tree with root $< a, b >$, then t/c will be the same tree, only with the root $< c, b >$.

Construction 2.3

(A) The set T_x of trees t for fixed x:

(1) The trees for minimal elements: For each U, $x \in \mu(U)$, we consider a tree existing by $(\mu\tau)$, i.e. for each Y s.t. $x \in Y$, we minimize, if necessary, and so on. We call such trees U,x-trees, and the set of such trees $T_{U,x}$.

(2) Construction of the set T'_x of trees for the nonminimal elements. Let $x \in Z$. Construct the tree t_x as follows (here, one tree per x suffices for all U):

Level 0: $< \emptyset, x >$

Level 1:

Choose arbitrary $f \in \Pi\{\mu(U) : x \in U \in \mathcal{Y}\}$. Note that $U \neq \emptyset \rightarrow \mu(U) \neq \emptyset$ by $Z = K$: This holds by the proof of Remark 2.4 (1), and the fact that $(\mu\tau)$ implies $(\mu Cum0)$ (see above Fact, (2)). By the same Fact, we can also use (μCum).

Let $\{< U, f(U) >: x \in U \in \mathcal{Y}\}$ be the set of children of $< \emptyset, x >$. This assures that the element will be nonminimal.

Level > 1:

Let $< U, f(U) >$ be an element of level 1, as $f(U) \in \mu(U)$, there is a $t_{U,f(U)} \in T\mu_{f(U)}$. Graft one of these trees $t_{U,f(U)} \in T\mu_{f(U)}$ at $< U, f(U) >$ on the level 1. This assures that a minimal element will be below it to guarantee smoothness.

Finally, let $T_x := T\mu_x \cup T'_x$.

(B) The relation $\lhd$ between trees:

For $x, y \in Z$, $t \in T_x$, $t' \in T_y$, set $t \rhd t'$ iff for some Y $< Y, y >$ is a child of the root $< X, x >$ in t, and t' is the subtree of t beginning at this $< Y, y >$.

(C) The structure $\mathcal{Z}$:

Let $\mathcal{Z} := < \{< x, t_x >: x \in Z, t_x \in T_x\}, < x, t_x > \succ < y, t_y >$ iff $t_x \rhd^* t_y >$.

The rest of the proof are simple observations.

Fact 2.14

(1) If $t_{U,x}$ is an U,x-tree, $< Y_n, y_n \in \overline{Y_n}, a >$ an element of $t_{U,x}$, $< Y_m, y_m \in \overline{Y_m}, a >$ a direct or indirect child of $< Y_n, y_n \in \overline{Y_n}, a >$, then $y_m \notin Y_n$.

(2) Let $< Y_n, y_n \in \overline{Y_n}, a >$ be an element in $t_{U,x} \in T\mu_x$, t' the subtree starting at $< Y_m, y_m \in \overline{Y_m}, a >$, then t' is a $Y_m, y_m - tree$.

(3) $\prec$ is free from cycles.

(4) If $t_{U,x}$ is an U,x-tree, then $< x, t_{U,x} >$ is $\prec -$minimal in $\mathcal{Z}\lceil U$.

(5) No $< x, t_x >$, $t_x \in T'_x$ is minimal in any $\mathcal{Z}\lceil U$, $U \in \mathcal{Y}$.

(6) Smoothness is respected for the elements of the form $< x, t_{U,x} >$.

(7) Smoothness is respected for the elements of the form $< x, t_x >$ with $t_x \in T'_x$.

(8) $\mu = \mu_{\mathcal{Z}}$.

Proof:

(1) trivial by (a) and (b).

(2) trivial by (a).

(3) Note that no $< x, t_x > t_x \in T'_x$ can be smaller than any other element (smaller elements require $U \neq \emptyset$ at the root). So no cycle involves any such $< x, t_x >$. Consider now $< x, t_{U,x} >$, $t_{U,x} \in T\mu_x$. For any $< y, t_{V,y} > \prec < x, t_{U,x} >$, $y \notin U$ by (1), but $x \in \mu(U) \subseteq U$, so $x \neq y$.

(4) This is trivial by (1).

(5) Let $x \in U \in \mathcal{Y}$, then the construction of level 1 of t_x chooses $y \in \mu(U) \neq \emptyset$, and some $< y, t_{U,y} >$ is in $\mathcal{Z}\lceil U$ and below $< x, t_x >$.

(6) Let $x \in A \in \mathcal{Y}$, we have to show that either $< x, t_{U,x} >$ is minimal in $\mathcal{Z}\lceil A$, or that there is $< y, t_y > \prec < x, t_{U,x} >$ minimal in $\mathcal{Z}\lceil A$.

Case 1, $A \subseteq U$: Then $< x, t_{U,x} >$ is minimal in $\mathcal{Z}\lceil A$, again by (1).

Case 2, $A \not\subseteq U$: Then A is one of the Y_1 considered for level 1. So there is $<< U, A >, f_1(A) >$ in level 1 with $f_1(A) \in \mu(A) \subseteq A$ by (ΔU). But note that by (1) all elements below $<< U, A >, f_1(A) >$ avoid $U \cup A$. Let t be the subtree of $t_{U,x}$ beginning at $<< U, A >, f_1(A) >$, then by (2) t is one of the $A, f_1(A) - trees$ (which avoids, in addition, U), and $< f_1(A), t >$ is minimal in $\mathcal{Z}\lceil U \cup A$ by (4), so in $\mathcal{Z}\lceil A$, and $< f_1(A), t > \prec < x, t_{U,x} >$.

(7) Let $x \in A \in \mathcal{Y}$, $< x, t_x >$, $t_x \in T'_x$, and consider the subtree t beginning at $< A, f(A) >$, then t is one of the $A, f(A)-$trees, and $< f(A), t >$ is minimal in $\mathcal{Z}\lceil A$ by (4).

(8) Let $x \in \mu(U)$. Then any $< x, t_{U,x} >$ is $\prec -$minimal in $\mathcal{Z}\lceil U$ by (4), so $x \in \mu_{\mathcal{Z}}(U)$. Conversely, let $x \in U - \mu(U)$. By (5), no $< x, t_x >$ is minimal in U. Consider now some

$< x, t_{V,x} >\in \mathcal{Z}$, so $x \in \mu(V)$. As $x \in U - \mu(U)$, $U \not\subseteq V$ by (ΔU). Thus U was considered in the construction of level 1 of $t_{V,x}$. Let t be the subtree of $t_{V,x}$ beginning at $< U, f\ (U) >$, avoiding also V, by (ΔU), and $f_1(U) \in \mu(U) \subseteq U$, and $< f_1(U), t >\prec< x, t_{V,x} >$.

$\square$ (Fact 2.14 and Proposition 2.13)

2.4 A remark on Arieli/Avron "General patterns . . . ."

We refer here to [AA00].

We have two consequence relations, $\vdash$ and $\mid\!\sim$.

The rules to consider are

LCC^n $\frac{\Gamma \mid\!\sim \psi_1, \Delta \ldots \Gamma \mid\!\sim \psi_n, \Delta \Gamma, \psi_1, \ldots, \psi_n \mid\!\sim \Delta}{\Gamma \mid\!\sim \Delta}$

RW^n $\frac{\Gamma \mid\!\sim \psi_i, \Delta i=1 \ldots n \Gamma, \psi_1, \ldots, \psi_n \vdash \phi}{\Gamma \mid\!\sim \phi, \Delta}$

Cum $\Gamma, \Delta \neq \emptyset$, $\Gamma \vdash \Delta \rightarrow \Gamma \mid\!\sim \Delta$

RM $\Gamma \mid\!\sim \Delta \rightarrow \Gamma \mid\!\sim \psi, \Delta$

CM $\frac{\Gamma \mid\!\sim \psi \Gamma \mid\!\sim \Delta}{\Gamma, \psi \mid\!\sim \Delta}$

s-R $\Gamma \cap \Delta \neq \emptyset \rightarrow \Gamma \mid\!\sim \Delta$

M $\Gamma \vdash \Delta$, $\Gamma \subseteq \Gamma'$, $\Delta \subseteq \Delta' \rightarrow \Gamma' \vdash \Delta'$

C $\frac{\Gamma_1 \vdash \psi, \Delta_1 \Gamma_2, \psi \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}$

Let $\mathcal{L}$ be any set. Define now $\Gamma \vdash \Delta$ iff $\Gamma \cap \Delta \neq \emptyset$. Then s-R and M for $\vdash$ are trivial. For C : If $\Gamma_1 \cap \Delta_1 \neq \emptyset$ or $\Gamma_1 \cap \Delta_1 \neq \emptyset$, the result is trivial. If not, $\psi \in \Gamma_1$ and $\psi \in \Delta_2$, which implies the result. So $\vdash$ is a scr. Consider now the rules for a sccr which is $\vdash -$plausible for this $\vdash$. Cum is equivalent to s-R, which is essentially (PlI) of Plausibility Logic. Consider RW^n. If ϕ is one of the ψ_i, then the consequence $\Gamma \mid\!\sim \phi, \Delta$ is a case of one of the other hypotheses. If not, $\phi \in \Gamma$, so $\Gamma \mid\!\sim \phi$ by s-R, so $\Gamma \mid\!\sim \phi, \Delta$ by RM (if Δ is finite). So, for this $\vdash$, RW^n is a consequence of s-R + RM.

We are left with LCC^n, RM, CM, s-R, it was shown in [Sch04] and [Sch96-3] that this does not suffice to guarantee smooth representability, by failure of $(\mu Cum1)$.

3 APPROXIMATION AND THE LIMIT VARIANT

3.1 Introduction

Distance based semantics give perhaps the clearest motivation for the limit variant. For instance, the Stalnaker/Lewis semantics for counterfactual conditionals defines $\phi > \psi$ to hold in a (classical) model m iff in those models of ϕ, which are closest to m, ψ holds. For this to make sense, we need, of course, a distance d on the model set. We call this approach the minimal variant. Usually, one makes a limit assumption: The set of $\phi-$models closest to m is not empty if ϕ is consistent - i.e. the $\phi-$models are not arranged around m in a way that they come closer and closer, without a minimal distance. This is, of course, a very strong assumption, and which is probably difficult to justify philosophically. It seems to have its only justification in the fact that it avoids degenerate cases, where, in above example, for consistent ϕ $m \models \phi > FALSE$ holds. As such, this assumption is unsatisfactory.

It is a natural and much more convincing solution to the problem to modify the basic definition, and work without this rather artificial assumption. We adopt what we call a "limit approach", and define $m \models \phi > \psi$ iff there is a distance d' such that for all $m' \models \phi$ and $d(m, m') \leq d'$ $m' \models \psi$. Thus, from a certain point onward, ψ becomes and stays true. We will call this definition the structural limit, as it is based directly on the structure (the distance on the model set).

The model sets to consider are spheres around m, $S := \{m' \in M(\phi) : d(m, m') \leq d'\}$ for some d', s.t. $S \neq \emptyset$. The system of such S is nested, i.e. totally ordered by inclusion; and if $m \models \phi$, it has a smallest element $\{m\}$, etc. When we forget the underlying structure, and consider just the properties of these systems of spheres around different m, and for different ϕ, we obtain what we call the algebraic limit.

The interest to investigate this algebraic limit is twofold: first, we shall see (for other kinds of structures) that there are reasonable and not so reasonable algebraic limits. Second, this distinction permits us to separate algebraic from logical problems, which have to do with definability of model sets, in short definability problems. We will see in the following section that we find common definability problems and also common solutions in the usual minimal, and the limit variant.

In particular, the decomposition into three layers on both sides (minimal and limit version) can reveal that a (seemingly) natural notion of structural limit results in algebraic properties which have not much to do any more with the minimal variant. So, to speak about a limit variant, we will demand that this variant is not only a natural structural limit, but results in a natural abstract limit, too. Conversely, if the algebraic limit preserves the properties of the minimal variant, there is hope that it preserves the logical properties, too - not more than hope, however, due to definability problems, see the next Section.

3.2 The algebraic limit

3.2.1 Discussion

There are basic problems with the algebraic limit in general preferential structures. A natural definition of the structural limit for preferential structures is the following: $\phi \mid\sim \psi$ iff there is an "initial segment" or "minimizing initial segment" S of the $\phi-$models, where ψ holds. An initial segment should have two properties: first, any $m \models \phi$ should be in S, or be minimized by some $m' \in S$ (i.e. $m' \prec m$), second, it should be downward closed, i.e. if $m \in S$ and $m' \prec m$, $m' \models \phi$, m' should also be in S. The first requirement generates a problem:

Example 3.1

Let $a \prec b$, $a \prec c$, $b \preceq d$, $c \prec d$ (but $\prec$ not transitive!), then $\{a, b\}$ and $\{a, c\}$ are such S and S', but there is no $S'' \subseteq S \cap S'$ which is an initial segment. If, for instance, in a and b ψ holds, in a and c ψ', then "in the limit" ψ and ψ' will hold, but not $\psi \wedge \psi'$. This does not seem right. We should not be obliged to give up ψ to obtain ψ'.

When we look at the system of such S generated by a preferential structure and its algebraic properties, we will therefore require it to be closed under finite intersections, or at least, that if S, S' are such segments, then there must be $S'' \subseteq S \cap S'$ which is also such a segment.

We make this official. Let $\Lambda(X)$ be the set of initial segments of X, then we require:

$(\Lambda\wedge)$ If $A, B \in \Lambda(X)$ then there is $C \subseteq A \cap B$, $C \in \Lambda(X)$.

More precisely, a limit should be a structural limit in a reasonable sense - whatever the underlying structure is -, and the resulting algebraic limit should respect $(\Lambda\wedge)$.

We should not demand too much, either. It would be wrong to demand closure under arbitrary intersections, as this would mean that there is an initial segment which makes all consequences true - trivializing the very idea of a limit.

But we can make our requirements more precise, and bind the limit variant closely to the minimal variant, by looking at the algebraic version of both.

Before we continue, we make the definitions of the limit versions of preferential and ranked preferential structures precise (the latter allows an important simplification of the former).

(3.2.2) Basic definitions and properties

Definition 3.1

(1) General preferential structures

(1.1) The version without copies:

Let $\mathcal{M} :=< U, \prec>$. Define

$Y \subseteq X \subseteq U$ is a minimizing initial segment, or MISE, of X iff:

(a) $\forall x \in X \exists x \in Y . y \preceq x$ - where $y \preceq x$ stands for $x \prec y$ or $x = y$

and

(b) $\forall y \in Y, \forall x \in X (x \prec y \rightarrow x \in Y)$.

(1.2) The version with copies:

Let $\mathcal{M} :=< \mathcal{U}, \prec>$ be as above. Define for $Y \subseteq X \subseteq \mathcal{U}$

Y is a minimizing initial segment, or MISE of X iff:

(a) $\forall < x, i > \in X \exists < y, j > \in Y . < y, j > \preceq < x, i >$

and

(b) $\forall < y, j > \in Y, \forall < x, i > \in X (< x, i > \prec < y, j > \rightarrow < x, i > \in Y)$.

(1.3) For $X \subseteq \mathcal{U}$, let $\Lambda(X)$ be the set of MISE of X.

(1.4) We say that a set $\mathcal{X}$ of MISE is cofinal in another set of MISE $\mathcal{X}'$ (for the same base set X) iff for all $Y' \in \mathcal{X}'$, there is $Y \in \mathcal{X}$, $Y \subseteq Y'$.

(1.5) A MISE X is called definable iff $\{x : \exists < x, i > \in X\} \in \boldsymbol{D}_{\mathcal{L}}$.

(1.6) $T \models_{\mathcal{M}} \phi$ iff there is $Y \in \Lambda(\mathcal{U} \lceil M(T))$ s.t. $Y \models \phi$. $\mathcal{U} \lceil M(T) := \{< x, i > \in \mathcal{U} : x \in M(T)\}$ - if there are no copies, we simplify in the obvious way.

(2) Ranked preferential structures

In the case of ranked structures, we may assume without loss of generality that the MISE sets have a particularly simple form:

For $X \subseteq U$ $A \subseteq X$ is MISE iff $X \neq \emptyset$ and $\forall a \in A \forall x \in X (x \prec a \vee x \perp a \rightarrow x \in A)$. (A is downward and horizontally closed.)

(3) Theory Revision

Recall that we have a distance d on the model set, and are interested in $y \in Y$ which are close to X.

Thus, given X,Y, we define analogously:

$B \subseteq Y$ is MISE iff

(1) $B \neq \emptyset$

(2) there is d' s.t. $B := \{y \in Y : \exists x \in X . d(x, y) \leq d'\}$ (we could also have chosen

$d(x,y) < d'$, this is not important).

And we define $\phi \in T * T'$ iff there is $B \in \Lambda(M(T), M(T'))$ $B \models \phi$.

Before we look at deeper problems, we show some basic facts about the algebraic properties.

Fact 3.1

(Taken from [Sch04].)

Let the relation $\prec$ be transitive. The following hold in the limit variant of general preferential structures:

(1) If $A \in \Lambda(Y)$, and $A \subseteq X \subseteq Y$, then $A \in \Lambda(X)$.

(2) If $A \in \Lambda(Y)$, and $A \subseteq X \subseteq Y$, and $B \in \Lambda(X)$, then $A \cap B \in \Lambda(Y)$.

(3) If $A \in \Lambda(Y)$, $B \in \Lambda(X)$, then there is $Z \subseteq A \cup B$ $Z \in \Lambda(Y \cup X)$.

The following hold in the limit variant of ranked structures without copies, where the domain is closed under finite unions and contains all finite sets.

(4) $A, B \in \Lambda(X) \rightarrow A \subseteq B$ or $B \subseteq A$,

(5) $A \in \Lambda(X)$, $Y \subseteq X$, $Y \cap A \neq \emptyset \rightarrow Y \cap A \in \Lambda(Y)$,

(6) $\Lambda' \subseteq \Lambda(X)$, $\bigcap \Lambda' \neq \emptyset \rightarrow \bigcap \Lambda' \in \Lambda(X)$.

(7) $X \subseteq Y$, $A \in \Lambda(X) \rightarrow \exists B \in \Lambda(Y).B \cap X = A$

Proof:

(1) trivial.

(2)

(2.1) $A \cap B$ is closed in Y : Let $<x,i> \in A \cap B$, $<y,j> \prec <x,i>$, then $<y,j> \in A$. If $<y,j> \notin X$, then $<y,j> \notin A$, *contradiction*. So $<y,j> \in X$, but then $<y,j> \in B$.

(2.2) $A \cap B$ minimizes Y : Let $<a,i> \in Y$.

(a) If $<a,i> \in A - B \subseteq X$, then there is $<y,j> \prec <a,i>$, $<y,j> \in B$. Xy closure of A, $<y,j> \in A$.

(b) If $<a,i> \notin A$, then there is $<a',i'> \in A \subseteq X$, $<a',i'> \prec <a,i>$, continue by (a).

(3)

Let $Z := \{<x,i> \in A$: $\neg\exists <b,j> \preceq <x,i> . <b,j> \in X - B\} \cup \{<y,j> \in B$: $\neg\exists <a,i> \preceq <y,j> . <a,i> \in Y - A\}$, where $\preceq$ stands for $\prec$ or $=$.

(3.1) Z minimizes $Y \cup X$: We consider Y, X is symmetrical.

(a) We first show: If $< a, k >\in A-$Z, then there is $< y, i >\in Z$. $< a, k >\succ< y, i >$. Broof: If $< a, k >\in A-$Z, then there is $< b, j >\preceq< a, k >$, $< b, j >\in X-$B. Then there is $< y, i >\prec< b, j >$, $< y, i >\in B$. Xut $< y, i >\in Z$, too: If not, there would be $< a', k' >\preceq< y, i >$, $< a', k' >\in Y-$A, but $< a', k' >\prec< a, k >$, contradicting closure of A.

(b) If $< a'', k'' >\in Y-$A, there is $< a, k >\in A$, $< a, k >\prec< a'', k'' >$. If $< a, k >\notin Z$, continue with (a).

(3.2) Z is closed in $Y \cup X$: Let then $< z, i >\in Z$, $< u, k >\prec< z, i >$, $< u, k >\in Y \cup X$. Suppose $< z, i >\in A$ - the case $< z, i >\in B$ is symmetrical.

(a) $< u, k >\in Y - A$ cannot be, by closure of A.

(b) $< u, k >\in X - B$ cannot be, as $< z, i >\in Z$, and by definition of Z.

(c) If $< u, k >\in A-$Z, then there is $< v, l >\preceq< u, k >$, $< v, l >\in X-$B, so $< v, l >\prec< z, i >$, contradicting (b).

(d) If $< u, k >\in B-$Z, then there is $< v, l >\preceq< u, k >$, $< v, l >\in Y-$A, contradicting (a).

(4) Suppose not, so there are $a \in A-$B, $b \in B-$A. But if $a \perp b$, $a \in B$ and $b \in A$, similarly if $a \prec b$ or $b \prec a$.

(5) As $A \in \Lambda(X)$ and $Y \subseteq X$, $Y \cap A$ is downward and horizontally closed. As $Y \cap A \neq \emptyset$, $Y \cap A$ minimizes Y.

(6) $\bigcap \Lambda'$ is downward and horizontally closed, as all $A \in \Lambda'$ are. As $\bigcap \Lambda' \neq \emptyset$, $\bigcap \Lambda'$ minimizes X.

(7) Set $B := \{b \in Y : \exists a \in A. a \perp b$ or $b \leq a\}$

□

We have as immediate consequence:

Fact 3.2

If $\prec$ is transitive, then in the limit variant hold:

(1) (AND) holds,

(2) (OR) holds,

Proof:

Let $\mathcal{Z}$ be the structure.

(1) Immediate by Fact 3.1, (2) - set $A = B$.

(2) Immediate by Fact 3.1, (3). $\square$

(3.2.3) Translation between minimal and limit variant

Our aim is to analyze the limit version more closely, in particular, to see criteria whether the much more complex limit version can be reduced to the simpler minimal variant.

The problem is not simple, as there are two sides which come into play, and sometimes we need both to cooperate to achieve a satisfactory translation.

The first component is what we call the "algebraic limit", i.e. we stipulate that the limit version should have properties which correspond to the algebraic properties of the minimal variant. An exact correspondence cannot always be achieved, and we give a translation which seems reasonable.

But once the translation is done, even if it is exact, there might still be problems linked to translation to logic.

A good example is the property $(\mu =)$ of ranked structures:

$(\mu =)$ $X \subseteq Y$, $\mu(Y) \cap X \neq \emptyset \rightarrow \mu(Y) \cap X = \mu(X)$

or its logical form

$(\mid\!\sim=)$ $T \vdash T'$, $Con(\overline{\overline{T'}}, T) \rightarrow \overline{\overline{T}} = \overline{\overline{\overline{T'}} \cup T}$.

$\mu(Y)$ or its analogue $\overline{\overline{T'}}$ (set $X := M(T)$, $Y := M(T')$) speak about the limit, the "ideal", and this, of course, is not what we have in the limit version. This version was intoduced precisely to avoid speaking about the ideal.

So, first, we have to translate $\mu(Y) \cap X \neq \emptyset$ to something else, and the natural candidate seems to be

$\forall B \in \Lambda(Y).B \cap X \neq \emptyset$.

In logical terms, we have replaced the set of consequences of Y by some $Th(B)$ where $T' \subseteq Th(B) \subseteq \overline{\overline{T'}}$. The conclusion can now be translated in a similar way to $\forall B \in \Lambda(Y).\exists A \in \Lambda(X).A \subseteq B \cap X$ and $\forall A \in \Lambda(X).\exists B \in \Lambda(Y).B \cap X \subseteq A$. The total translation reads now:

$(\Lambda =)$ Let $X \subseteq Y$. Then $\forall B \in \Lambda(Y).B \cap X \neq \emptyset \rightarrow \forall B \in \Lambda(Y).\exists A \in \Lambda(X).A \subseteq B \cap X$ and $\forall A \in \Lambda(X).\exists B \in \Lambda(Y).B \cap X \subseteq A$.

By Fact 3.1 (5) and (7), we see that this holds in ranked structures. Thus, the limit reading seems to provide a correct algebraic limit.

Yet, Example 3.2 below shows the following:

Let $m' \neq m$ be arbitrary. For $T' := Th(\{m, m'\}), T := \emptyset$, we have $T' \vdash T$, $\overline{\overline{T'}} = Th(\{m'\})$, $\overline{\overline{T}} = Th(\{m\})$, $Con(\overline{\overline{T}}, T')$, but $Th(\{m'\}) = \overline{\overline{\overline{T'}} \cup T} \neq \overline{\overline{T}}$.

Thus:

(1) The prerequisite holds, though usually for $A \in \Lambda(T)$, $A \cap M(T') = \emptyset$. (2) (PR) fails, which is independent of the prerequisite $Con(\overline{\overline{T}}, T')$, so the problem is not just due to the prerequsite.

(3) Both inclusions fail.

We will see below in Corollary 4.6 a sufficient condition to make ($\mid\!\sim=$) hold in ranked structures. It has to do with definability or formulas, more precisely, the crucial property is to have sufficiently often $\overbrace{A} \cap \overbrace{M(T')} = \overbrace{A \cap M(T')}$ for $A \in \Lambda(T)$.

Example 3.2

(Taken from [Sch04].)

Take an infinite propositional language $p_i : i \in \omega$. We have ω_1 models (assume for simplicity CH).

Take the model m which makes all p_i true, and put it on top. Next, going down, take all models which make p_0 false, and then all models which make p_0 true, but p_1 false, etc. in a ranked construction. So, successively more p_i will become (and stay) true. Consequently, $\emptyset \models_\Lambda p_i$ for all i. But the structure has no minimum, and the "logical" limit m is not in the set wise limit. Let $T := \emptyset$ and $m' \neq m$, $T' := Th(\{m, m'\})$, then $\overline{\overline{T}} = Th(\{m\})$, $\overline{\overline{T'}} = Th(\{m'\})$, and $\overline{\overline{\overline{T'}} \cup T} = \overline{\overline{T'}} = Th(\{m'\})$.

This example shows that our translation is not perfect, but it is half the way. Note that the minimal variant faces the same problems (definability and others), so the problems are probably at least not totally due to our perhaps insufficient translation.

We turn to other rules.

($\Lambda\wedge$) If $A, B \in \Lambda(X)$ then there is $C \subseteq A \cap B$, $C \in \Lambda(X)$

seems a minimal requirement for an appropriate limit. It holds in transitive structures by Fact 3.2 (1).

The central logical condition for minimal smooth structures is

(CUM) $T \subseteq T' \subseteq \overline{\overline{T}} \rightarrow \overline{\overline{T}} = \overline{\overline{T'}}$

It would again be wrong - using the limit - to translate this only partly by: If $T \subseteq T' \subseteq \overline{\overline{T}}$, then for all $A \in \Lambda(M(T))$ there is $B \in \Lambda(M(T'))$ s.t. $A \subseteq B$ - and vice versa. Now, smoothness is in itself a wrong condition for limit structures, as it speaks about minimal elements, which we will not necessarily have. This cannot guide us. But when we consider a more modest version of cumulativity, we see what to do.

(CUMfin) If $T \mid\sim \phi$, then $\overline{\overline{T}} = \overline{\overline{T \cup \{\phi\}}}$.

This translates into algebraic limit conditions as follows - where $Y = M(T)$, and $X = M(T \cup \{\phi\})$.

$(\Lambda CUMfin)$ Let $X \subseteq Y$. If there is $B \in \Lambda(Y)$ s.t. $B \subseteq X$, then: $\forall A \in \Lambda(X) \exists B' \in \Lambda(Y).B' \subseteq A$ and $\forall B' \in \Lambda(Y) \exists A \in \Lambda(X).A \subseteq B'$.

Note, that in this version, we do not have the "ideal" limit on the left of the implication, but one fixed approximation $B \in \Lambda(Y)$. We can now prove that $(\Lambda CUMfin)$ holds in transitive structures: The first part holds by Fact 3.1 (2), the second, as $B \cap B' \in \Lambda(Y)$ by Fact 3.1 (1). This is true without additional properties of the structure, which might at first sight be surprising. But note that the initial segments play a similar role as the set of minimal elements: an initial segment has to minimize the other elements, as the set of minimal elements in the smooth case.

The central algebraic property of minimal preferential structures is

(μPR) $X \subseteq Y \to \mu(Y) \cap X \subseteq \mu(X)$

This translates naturally and directly to

(ΛPR) $X \subseteq Y \to \forall A \in \Lambda(X) \exists B \in \Lambda(Y).B \cap X \subseteq A$

(ΛPR) holds in transitive structures: $Y - X \in \Lambda(Y - X)$, so the result holds by Fact 3.1 (3).

The central algebraic condition of ranked minimal structures is

$(\mu =)$ $X \subseteq Y$, $\mu(Y) \cap X \neq \emptyset \to \mu(Y) \cap X = \mu(X)$

We saw above to translate this condition to $(\Lambda =)$, we also saw that $(\Lambda =)$ holds in ranked structures.

We will see in Section 4, Corollary 4.6 that the following logical version holds in raked structures:

$T \not\mid\sim \neg\gamma$ implies $\overline{\overline{T}} = \overline{\overline{T \cup \{\gamma\}}}$

We generalize above results to a receipt:

Translate

- $\mu(X) \subseteq \mu(Y)$ to $\forall B \in \Lambda(Y) \exists A \in \Lambda(X).A \subseteq B$, and thus

- $\mu(Y) \cap X \subseteq \mu(X)$ to $\forall A \in \Lambda(X) \exists B \in \Lambda(Y).B \cap X \subseteq A$,

- $\mu(X) \subseteq Y$ to $\exists A \in \Lambda(X).A \subseteq Y$, and thus

- $\mu(Y) \cap X \neq \emptyset$ to $\forall B \in \Lambda(Y).B \cap X \neq \emptyset$

- $X \subseteq \mu(Y)$ to $\forall B \in \Lambda(Y).X \subseteq B$,

and quantify expressions separately, thus we repeat:

- (μCUM) $\mu(Y) \subseteq X \subseteq Y \rightarrow \mu(X) = \mu(Y)$ translates to

$(\Lambda CUMfin)$ Let $X \subseteq Y$. If there is $B \in \Lambda(Y)$ s.t. $B \subseteq X$, then: $\forall A \in \Lambda(X) \exists B' \in \Lambda(Y).B' \subseteq A$ and $\forall B' \in \Lambda(Y) \exists A \in \Lambda(X).A \subseteq B'$.

- $(\mu =)$ $X \subseteq Y$, $\mu(Y) \cap X \neq \emptyset \rightarrow \mu(Y) \cap X = \mu(X)$ translates to

$(\Lambda =)$ Let $X \subseteq Y$. If $\forall B \in \Lambda(Y).B \cap X \neq \emptyset$, then $\forall A \in \Lambda(X) \exists B' \in \Lambda(Y).B' \cap X \subseteq A$, and $\forall B' \in \Lambda(Y) \exists A \in \Lambda(X).A \subseteq B' \cap X$.

We collect now for easier reference the definitions and some algebraic properties which we saw above to hold:

Definition 3.2

$(\Lambda\wedge)$ If $A, B \in \Lambda(X)$ then there is $C \subseteq A \cap B$, $C \in \Lambda(X)$,

(ΛPR) $X \subseteq Y \rightarrow \forall A \in \Lambda(X) \exists B \in \Lambda(Y).B \cap X \subseteq A$,

$(\Lambda CUMfin)$ Let $X \subseteq Y$. If there is $B \in \Lambda(Y)$ s.t. $B \subseteq X$, then: $\forall A \in \Lambda(X) \exists B' \in \Lambda(Y).B' \subseteq A$ and $\forall B' \in \Lambda(Y) \exists A \in \Lambda(X).A \subseteq B'$,

$(\Lambda =)$ Let $X \subseteq Y$. Then $\forall B \in \Lambda(Y).B \cap X \neq \emptyset \rightarrow \forall B \in \Lambda(Y).\exists A \in \Lambda(X).A \subseteq B \cap X$ and $\forall A \in \Lambda(X).\exists B \in \Lambda(Y).B \cap X \subseteq A$.

Fact 3.3

In transitive structures hold:

(1) $(\Lambda\wedge)$

(2) (ΛPR)

(3) $(\Lambda CUMfin)$

In ranked structures holds:

(4) $(\Lambda =)$

Proof:

(1) By Fact 3.2 (1).

(2) $Y - X \in \Lambda(Y - X)$, so the result holds by Fact 3.1 (3).

(3) By Fact 3.1 (1) and (2).

(4) By Fact 3.1 (5) and (7).

□

To summarize:

Just as in the minimal case, the algebraic laws may hold, but not the logical ones, due in both cases to definability problems. Thus, we cannot expect a clean proof of correspondence. But we can argue that we did a correct translation, which shows its limitation, too. The part with $\mu(X)$ and $\mu(Y)$ on both sides of $\subseteq$ is obvious, we will have a perfect correspondence. The part with $X \subseteq \mu(Y)$ is obvious, too. The problem is in the part with $\mu(X) \subseteq Y$. As we cannot use the limit, but only its approximation, we are limited here to one (or finitely many) consequences of T, if $X = M(T)$, so we obtain only $T \mid\!\sim \phi$, if $Y \subseteq M(\phi)$, and if there is $A \in \Lambda(X).A \subseteq Y$.

We consider a limit only appropriate, if it is an algebraic limit which preserves algebraic properties of the minimal version in above translation.

The advantage of such limits is that they allow - with suitable caveats - to show that they preserve the logical properties of the minimal variant, and thus are equivalent to the minimal case (with, of course, perhaps a different relation). Thus, they allow a straightforward trivialization.

3.3 Booth revision - approximation by formulas

Booth and his co-authors [...] have shown in a very interesting paper that many new approaches to theory revision (with fixed K) can be represented by two relations, $<$ and $\lhd$, where $<$ is the usual ranked relation, and $\lhd$ is a sub-relation of $<$. They have, however, left open a characterization of the infinite case, which we will treat here. Our approach is basically semantic, though we use sometimes the language of logic, on the one hand to show how to approximate with formulas a single model, and on the other hand when we use classical compactness. This is, however, just a matter of speaking, and we could translate it into model sets, too, but we do not think that we would win much by doing so. Moreoever, we will treat only the formula case, as this seems to be the most interesting (otherwise the problem of approximation by formulas would not exist), and restrict ourselves to the definability preserving case. The more general case is left open, for a young researcher who wants to sharpen his tools by solving it. Another open problem is to treat the same question for variable K, for distance based revision.

We change perspective a little, and work directly with a ranked relation, so we forget about the (fixed) K of revision, and have an equivalent, ranked structure. We are then interested in an operator ν, which returns a model set $\nu(\phi) := \nu(M(\phi))$, where $\nu(\phi) \cap M(\phi)$ is given by a ranked relation $<$, and $\nu(\phi) - M(\phi) := \{x \notin M(\phi) : \exists y \in \nu(\phi) \cap M(\phi)(x \lhd y)\}$, and $\lhd$ is an arbitrary subrelation of $<$. The essential problem is to find such y, as we have only formulas to find it. (If we had full theories, we could just look at all $Th(\{y\})$ whether $x \in \nu(Th(\{y\}))$.) There is still some more work to do, as we have to connect the two relations, and simply taking a ready representation result will not do, as we shall see.

We first introduce some notation, then a set of conditions, and formulate the representation result. Soundness will be trivial. For completeness, we construct first the ranked relation $<$, show that it does what it should do, and then the subrelation $\lhd$.

For fundamentals, the reader is referred to Section 4.3, where we treat the ranked case more systematically.

Notation 3.1

We set

$\mu^+(X) := \nu(X) \cap X$

$\mu^-(X) := \nu(X) - X$

where $X := M(\phi)$ for some ϕ.

Condition 3.1

(μ^-1) $Y \cap \mu^-(X) \neq \emptyset \rightarrow \mu^+(Y) \cap X = \emptyset$

(μ^-2) $Y \cap \mu^-(X) \neq \emptyset \rightarrow \mu^+(X \cup Y) = \mu^+(Y)$

(μ^-3) $Y \cap \mu^-(X) \neq \emptyset \rightarrow \mu^-(Y) \cap X = \emptyset$

(μ^-4) $\mu^+(A) \subseteq \mu^+(B) \rightarrow \mu^-(A) \subseteq \mu^-(B)$

(μ^-5) $\mu^+(X \cup Y) = \mu^+(X) \cup \mu^+(Y) \rightarrow \mu^-(X \cup Y) = \mu^-(X) \cup \mu^-(Y)$

Fact 3.4

(μ^-1) and $(\mu\emptyset)$, $(\mu \subseteq)$ for μ^+ imply

(1) $\mu^+(X) \cap Y \neq \emptyset \rightarrow \mu^+(X) \cap \mu^-(Y) = \emptyset$

(2) $X \cap \mu^-(X) = \emptyset$.

Proof:

(1) Let $\mu^+(X) \cap \mu^-(Y) \neq \emptyset$, then $X \cap \mu^-(Y) \neq \emptyset$, so by (μ^-1) $\mu^+(X) \cap Y = \emptyset$.

(2) Set $X := Y$, and use $(\mu\emptyset)$, $(\mu \subseteq)$, (μ^-1), (1).

$\square$

Proposition 3.5

$\nu : \{M(\phi) : \phi \in F(\mathcal{L})\} \rightarrow \boldsymbol{D}_{\mathcal{L}}$ is representable by $<$ and $\lhd$, where $<$ is a smooth ranked relation, and $\lhd$ a subrelation of $<$, and $\mu^+(X)$ is the usual set of $<-$minimal elements of

X, and $\mu^-(X) = \{x \notin X : \exists y \in \mu^+(X).(x \lhd y)\}$, iff the following conditions hold: $(\mu \subseteq)$, $(\mu\emptyset)$, $(\mu =)$ for μ^+, and $(\mu^-1) - (\mu^-5)$ for μ^+ and μ^-.

Proof:

Soundness:

The first three hold for smooth ranked structures, and the others are easily verified.

Completeness:

(A) We first show how to generate the ranked relation $<$.

There is a small problem.

The author first thought that one may take any result for ranked structures off the shelf, plug in the other relation somehow (see the second half), and thats it. No, that *isn't* it: Suppose there is x, and a sequence x_i converging to x in the usual topology. Thus, if $x \in M(\phi)$, then there will always be some x_i in $M(\phi)$, too. Take now a ranked structure $\mathcal{Z}$, where all the x_i are strictly smaller than x. Consider $\mu(\phi)$, this will usually not contain x (avoid some nasty things with definability), so in the usual construction ($\preceq_1$ below), x will not be forced to be below any element y, how high up $y > x$ might be. However, there is ψ separating x and y, e.g. $x \models \neg\psi$, $y \models \psi$, and if we take as the second relation just the ranking again, $x \in \mu^-(\psi)$, so this becomes visible.

Consequently, considering μ^- may give strictly more information, and we have to put in a little more work. We just patch a proof for simple ranked structures, adding information obtained through μ^-.

We follow closely the strategy of the proof of 3.10.11 in [Sch04]. We will, however, change notation at one point: the relation R in [Sch04] is called $\preceq$ here. The proof goes over several steps, which we will enumerate.

Note that by Fact 4.11 (2)+(3)+(4) below, taken from [Sch04], $(\mu \parallel)$, $(\mu\cup)$, $(\mu\cup')$, $(\mu =')$ hold, as the prerequisites about the domain are valid.

(1) To generate the ranked relation $<$, we define two relations, $\preceq_1$ and $\preceq_2$, where $\preceq_1$ is the usual one for ranked structures, as defined in the proof of 3.10.11 of [Sch04], $a \preceq_1 b$ iff $a \in \mu^+(X)$, $b \in X$, or $a = b$, and $a \preceq_2 b$ iff $a \in \mu^-(X)$, $b \in X$.

Moreover, we set $a \preceq b$ iff $a \preceq_1 b$ or $a \preceq_2 b$.

(2) Obviously, $\preceq$ is reflexive, we show that $\preceq$ is transitive by looking at the four different cases.

(2.1) In [Sch04], it was shown that $a \preceq_1 b \preceq_1 c \rightarrow a \preceq_1 c$. For completeness' sake, we repeat the argument: Suppose $a \preceq_1 b$, $b \preceq_1 c$, let $a \in \mu^+(A)$, $b \in A$, $b \in \mu^+(B)$, $c \in B$. We show $a \in \mu^+(A \cup B)$. By $(\mu \parallel)$ $a \in \mu^+(A \cup B)$ or $b \in \mu^+(A \cup B)$. Suppose $b \in \mu^+(A \cup B)$, then $\mu^+(A \cup B) \cap A \neq \emptyset$, so by $(\mu =)$ $\mu^+(A \cup B) \cap A = \mu^+(A)$, so $a \in \mu^+(A \cup B)$.

(2.2) Suppose $a \preceq_1 b \preceq_2 c$, we show $a \preceq_1 c$: Let $c \in Y$, $b \in \mu^-(Y) \cap X$, $a \in \mu^+(X)$. Consider $X \cup Y$. As $X \cap \mu^-(Y) \neq \emptyset$, by $(\mu^- 2)$ $\mu^+(X \cup Y) = \mu^+(X)$, so $a \in \mu^+(X \cup Y)$ and $c \in X \cup Y$, so $a \preceq_1 c$.

(2.3) Suppose $a \preceq_2 b \preceq_2 c$, we show $a \preceq_2 c$: Let $c \in Y$, $b \in \mu^-(Y) \cap X$, $a \in \mu^-(X)$. Consider $X \cup Y$. As $X \cap \mu^-(Y) \neq \emptyset$, by $(\mu^- 2)$ $\mu^+(X \cup Y) = \mu^+(X)$, so by $(\mu^- 5)$ $\mu^-(X \cup Y) = \mu^-(X)$, so $a \in \mu^-(X \cup Y)$ and $c \in X \cup Y$, so $a \preceq_2 c$.

(2.4) Suppose $a \preceq_2 b \preceq_1 c$, we show $a \preceq_2 c$: Let $c \in Y$, $b \in \mu^+(Y) \cap X$, $a \in \mu^-(X)$. Consider $X \cup Y$. As $\mu^+(Y) \cap X \neq \emptyset$, $\mu^+(X) \subseteq \mu^+(X \cup Y)$. (Here is the argument: By $(\mu \parallel)$, $\mu^+(X \cup Y) = \mu^+(X) \parallel \mu^+(Y)$, so, if $\mu^+(X) \not\subseteq \mu^+(X \cup Y)$, then $\mu^+(X) \cap \mu^+(X \cup Y) = \emptyset$, so $\mu^+(X) \cap (X \cup Y - \mu^+(X \cup Y)) \neq \emptyset$ by $(\mu\emptyset)$, so by $(\mu\cup')$ $\mu^+(X \cup Y) = \mu^+(Y)$. But if $\mu^+(Y) \cap X = \mu^+(X \cup Y) \cap X \neq \emptyset$, $\mu^+(X) = \mu^+(X \cup Y) \cap X$ by $(\mu =)$, so $\mu^+(X) \cap \mu^+(X \cup Y) \neq \emptyset$, *contradiction.*) So $\mu^-(X) \subseteq \mu^-(X \cup Y)$ by $(\mu^- 4)$, so $c \in X \cup Y$, $a \in \mu^-(X \cup Y)$, and $a \preceq_2 c$.

(3) We also see:

(3.1) $a \in \mu^+(A)$, $b \in A - \mu^+(A) \rightarrow b \not\preceq a$.

(3.2) $a \in \mu^-(A)$, $b \in A \rightarrow b \not\preceq a$.

Proof of (3.1):

(a) $\neg(b \preceq_1 a)$ was shown in [Sch04], we repeat again the argument: Suppose there is B s.t. $b \in \mu^+(B)$, $a \in B$. Then by $(\mu\cup)$ $\mu^+(A \cup B) \cap B = \emptyset$, and by $(\mu\cup')$ $\mu^+(A \cup B) = \mu^+(A)$, but $a \in \mu^+(A) \cap B$, *contradiction.*

(b) Suppose there is B s.t. $a \in B$, $b \in \mu^-(B)$. But $A \cap \mu^-(B) \neq \emptyset$ implies $\mu^+(A) \cap B = \emptyset$ by $(\mu^- 1)$.

Proof of (3.2):

(a) Suppose $b \preceq_1 a$, so there is B s.t. $a \in B$, $b \in \mu^+(B)$, so $B \cap \mu^-(A) \neq \emptyset$, so $\mu^+(B) \cap A = \emptyset$ by $(\mu^- 1)$. - - (b) Suppose $b \preceq_2 a$, so there is B s.t. $a \in B$, $b \in \mu$ (B), so $B \cap \mu$ $(A) \neq \emptyset$, so $\mu^-(B) \cap A = \emptyset$ by $(\mu^- 3)$.

(4) Let by Lemma 3.10.7 in [Sch04] S be a total, transitive, reflexive relation on U which extends $\preceq$ s.t. xSy,ySx $\rightarrow x \preceq y$ (recall that $\preceq$ is transitive and reflexive). Define $a < b$ iff aSb, but not bSa. If $a \perp b$ (i.e. neither $a < b$ nor $b < a$), then, by totality of S, aSb and bSa. $<$ is ranked: If $c < a \perp b$, then by transitivity of S cSb, but if bSc, then again by transitivity of S aSc. Similarly for $c > a \perp b$.

(5) It remains to show that $<$ represents μ and is $\mathcal{Y}-$smooth:

Let $a \in A - \mu^+(A)$. By $(\mu\emptyset)$, $\exists b \in \mu^+(A)$, so $b \preceq_1 a$, but by case (3.1) above $a \not\preceq b$, so bSa, but not aSb, so $b < a$, so $a \in A - \mu_<(A)$. Let $a \in \mu^+(A)$, then for all $a' \in A$ $a \preceq a'$, so aSa', so there is no $a' \in A$ $a' < a$, so $a \in \mu_<(A)$. Finally, $\mu^+(A) \neq \emptyset$, all $x \in \mu^+(A)$ are minimal in A as we just saw, and for $a \in A - \mu^+(A)$ there is $b \in \mu^+(A)$, $b \preceq_1 a$, so the structure is smooth.

(B) The subrelation $\lhd$:

Let $x \in \mu^-(X)$, we look for $y \in \mu^+(X)$ s.t. $x \lhd y$ where $\lhd$ is the smaller, additional relation. By the definition of the relation $\preceq_2$ above, we know that $\lhd \subseteq \preceq$ and by (3.2) above $\lhd \subseteq <$.

Take an arbitrary enumeration of the propositional variables of $\mathcal{L}$, $p_i : i < \kappa$. We will inductively decide for p_i or $\neg p_i$. σ etc. will denote a finite subsequence of the choices made so far, i.e. $\sigma = \pm p_{i_0}, \ldots, \pm p_{i_n}$ for some $n < \omega$. Given such σ, $M(\sigma) := M(\pm p_{i_0}) \cap \ldots \cap M(\pm p_{i_n})$. $\sigma + \sigma'$ will be the union of two such sequences, this is again one such sequence.

Take an arbitrary model m for $\mathcal{L}$, i.e. a function $m : v(\mathcal{L}) \to \{t, f\}$. We will use this model as a "strategy", which will tell us how to decide, if we have some choice.

We determine y by an inductive process, essentially cutting away $\mu^+(X)$ around y. We choose p_i or $\neg p_i$ preserving the following conditions inductively: For all finite sequences σ as above we have:

(1) $M(\sigma) \cap \mu^+(X) \neq \emptyset$,

(2) $x \in \mu^-(X \cap M(\sigma))$.

For didactic reasons, we do the case p_0 separately.

Consider p_0. Either $M(p_0) \cap \mu^+(X) \neq \emptyset$, or $M(\neg p_0) \cap \mu^+(X) \neq \emptyset$, or both. If e.g. $M(p_0) \cap \mu^+(X) \neq \emptyset$, but $M(\neg p_0) \cap \mu^+(X) = \emptyset$, then we have no choice, and we take p_0, in the opposite case, we take $\neg p_0$. E.g. in the first case, $\mu^+(X \cap M(p_0)) = \mu^+(X)$, so $x \in \mu^-(X \cap M(p_0))$ by (μ^-4). If both intersections are non-empty, then by (μ^-5) $x \in \mu^-(X \cap M(p_0))$ or $x \in \mu^-(X \cap M(\neg p_0))$, or both. Only in the last case, we use our strategy to decide whether to choose p_0 or $\neg p_0$: if $m(p_0) = t$, we choose p_0, if not, we choose $\neg p_0$.

Obviously, (1) and (2) above are satisfied.

Suppose we have chosen p_i or $\neg p_i$ for all $i < \alpha$, i.e. defined a partial function from $v(\mathcal{L})$ to $\{t, f\}$, and the induction hypotheses (1) and (2) hold. Consider p_α. If there is no finite subsequence σ of the choices done so far s.t. $M(\sigma) \cap M(p_\alpha) \cap \mu^+(X) = \emptyset$, then p_α is a candidate. Likewise for $\neg p_\alpha$.

One of p_α or $\neg p_\alpha$ is a candidate:

Suppose not, then there are σ and σ' subsequences of the choices done so far, and $M(\sigma) \cap M(p_\alpha) \cap \mu^+(X) = \emptyset$ and $M(\sigma') \cap M(\neg p_\alpha) \cap \mu^+(X) = \emptyset$. But then $M(\sigma + \sigma') \cap \mu^+(X) = M(\sigma) \cap M(\sigma') \cap \mu^+(X) \subseteq M(\sigma) \cap M(p_\alpha) \cap \mu^+(X) \cup M(\sigma') \cap M(\neg p_\alpha) \cap \mu^+(X) = \emptyset$, contradicting (1) of the induction hypothesis.

So induction hypothesis (1) will hold again.

Recall that for each candidate and any σ by induction hypothesis (1) $M(\sigma) \cap M(p_\alpha) \cap \mu^+(X) = \mu^+(M(\sigma) \cap M(p_\alpha) \cap X)$ by $(\mu =')$, and also for $\sigma \subseteq \sigma'$ $\mu^+(M(\sigma') \cap M(p_\alpha) \cap X) \subseteq \mu^+(M(\sigma) \cap M(p_\alpha) \cap X)$ by $(\mu =')$ and $M(\sigma') \subseteq M(\sigma)$, and thus by $(\mu^- 4)$ $\mu^-(M(\sigma') \cap M(p_\alpha) \cap X) \subseteq \mu^-(M(\sigma) \cap M(p_\alpha) \cap X)$.

If we have only one candidate left, say e.g. p_α, then for each sufficiently big sequence σ $M(\sigma) \cap M(\neg p_\alpha) \cap \mu^+(X) = \emptyset$, thus for such σ $\mu^+(M(\sigma) \cap M(p_\alpha) \cap X) = M(\sigma) \cap M(p_\alpha) \cap \mu^+(X) = M(\sigma) \cap \mu^+(X) = \mu^+(M(\sigma) \cap X)$, and thus by $(\mu^- 4)$ $\mu^-(M(\sigma) \cap M(p_\alpha) \cap X) = \mu^-(M(\sigma) \cap X)$, so $\neg p_\alpha$ plays no really important role. In particular, induction hypothesis (2) holds again.

Suppose now that we have two candidates, thus for p_α and $\neg p_\alpha$ and each σ $M(\sigma) \cap M(p_\alpha) \cap \mu^+(X) \neq \emptyset$ and $M(\sigma) \cap M(\neg p_\alpha) \cap \mu^+(X) \neq \emptyset$.

By the same kind of argument as above we see that either for p_α or for $\neg p_\alpha$, or for both, and for all σ $x \in \mu^-(M(\sigma) \cap M(p_\alpha) \cap X)$ or $x \in \mu^-(M(\sigma) \cap M(\neg p_\alpha) \cap X)$.

If not, there are σ and σ' and $x \notin \mu^-(M(\sigma) \cap M(p_\alpha) \cap X) \supseteq \mu^-(M(\sigma + \sigma') \cap M(p_\alpha) \cap X)$ and $x \notin \mu^-(M(\sigma') \cap M(\neg p_\alpha) \cap X) \supseteq \mu^-(M(\sigma + \sigma') \cap M(\neg p_\alpha) \cap X)$, but $\mu^-(M(\sigma + \sigma') \cap X) = \mu^-(M(\sigma + \sigma') \cap M(p_\alpha) \cap X) \cup \mu^-(M(\sigma + \sigma') \cap M(\neg p_\alpha) \cap X)$, so $x \notin \mu^-(M(\sigma + \sigma') \cap X)$, contradicting the induction hypothesis (2).

If we can choose both, we let the strategy decide, as for p_0.

So induction hypotheses (1) and (2) will hold again.

This gives a complete description of some y (relative to the strategy!), and we set $x \lhd y$. We have to show: for all $Y \in \mathcal{Y}$ $x \in \mu^-(Y) \leftrightarrow x \in \mu_\lhd(Y) :\leftrightarrow \exists y \in \mu^+(Y).x \lhd y$. As we will do above construction for all Y, it suffices to show that $y \in \mu^+(X)$ for "$\rightarrow$". Conversely, if the y constructed above is in $\mu^+(Y)$, then x has to be in $\mu^-(Y)$ for "$\leftarrow$".

If $y \notin \mu^+(X)$, then $Th(y)$ is inconsistent with $Th(\mu^+(X))$, as μ^+ is definability preserving, so by classical compactness there is a suitable finite sequence σ with $M(\sigma) \cap \mu^+(X) = \emptyset$, but this was excluded by the induction hypothesis (1). So $y \in \mu^+(X)$.

Suppose $y \in \mu^+(Y)$, but $x \notin \mu^-(Y)$. So $y \in \mu^+(Y)$ and $y \in \mu^+(X)$, and $Y = M(\phi)$ for some ϕ, so there will be a suitable finite sequence σ s.t. for all σ' with $\sigma \subseteq \sigma'$ $M(\sigma') \cap X \subseteq M(\phi) = Y$, and by our construction $x \in \mu^-(M(\sigma') \cap X)$. As $y \in \mu^+(X) \cap \mu^+(Y) \cap (M(\sigma') \cap X)$, $\mu^+(M(\sigma') \cap X) \subseteq \mu^+(Y)$, so by $(\mu^- 4)$ $\mu^-(M(\sigma') \cap X) \subseteq \mu^-(Y)$, so $x \in \mu^-(Y)$, *contradiction.*

We do now this construction for all strategies. Obviously, this does not modify our results.

This finishes the completeness proof. $\square$

As we postulated definability preservation, there are no problems to translate the result into logic. (Note that ν was applied to formula defined model sets, but the resulting sets were perhaps theory defined model sets.)

4 DEFINABILITY PRESERVATION

4.1 General remarks, affected conditions

We assume now $\mathcal{Y} \subseteq \mathcal{P}(Z)$ to be closed under arbitrary intersections (this is used for the definition of $\widehat{\cdot}$) and finite unions, and $\emptyset, Z \in \mathcal{Y}$. This holds, of course, for $\mathcal{Y} = \boldsymbol{D}_{\mathcal{L}}$, $\mathcal{L}$ any propositional language.

The aim of Sections 4.1 and 4.2 is to present the results of [Sch04] connected to problems of definability preservation in a uniform way, stressing the crucial condition $\widehat{X} \cap \widehat{Y} = \widehat{X \cap Y}$. This presentation shall help and guide future research concerning similar problems.

For motivation, we first consider the problem with definability preservation for the rules

(PR) $\overline{\overline{T \cup T'}} \subseteq \overline{\overline{\overline{T}} \cup T'}$, and

($\mid\sim=$) $T \vdash T'$, $Con(\overline{\overline{T'}}, T) \rightarrow \overline{\overline{T}} = \overline{\overline{\overline{T'}} \cup T}$ holds.

which are consequences of

(μPR) $X \subseteq Y \rightarrow \mu(Y) \cap X \subseteq \mu(X)$ or

$(\mu =)$ $X \subseteq Y$, $\mu(Y) \cap X \neq \emptyset \rightarrow \mu(Y) \cap X = \mu(X)$ respectively

and definability preservation.

First, in the general case without definability preservation, (PR) fails, and in the ranked case, ($\mid\sim=$) may fail. So failure is not just a consequence of the very liberal definition of general preferential structures.

Example 4.1

(1) This example was first given in [Sch92].

Let $v(\mathcal{L}) := \{p_i : i \in \omega\}$, $n, n' \in M_{\mathcal{L}}$ be defined by $n \models \{p_i : i \in \omega\}$, $n' \models \{\neg p_0\} \cup \{p_i : 0 < i < \omega\}$. Let $\mathcal{M} :=< M_{\mathcal{L}}, \prec>$ where only $n \prec n'$, i.e. just two models are comparable. Let $\mu := \mu_{\mathcal{M}}$, and $\mid\sim$ be defined as usual by μ.

Set $T := \emptyset$, $T' := \{p_i : 0 < i < \omega\}$. We have $M_T = M_{\mathcal{L}}$, $\mu(M_T) = M_{\mathcal{L}} - \{n'\}$, $M_{T'} = \{n, n'\}$, $\mu(M_{T'}) = \{n\}$. So by Example 1.1, $\mathcal{M}$ is not definability preserving, and, furthermore, $\overline{\overline{T}} = \overline{T}$, $\overline{\overline{T'}} = \overline{\{p_i : i < \omega\}}$, *so* $p_0 \in \overline{\overline{T \cup T'}}$, *but* $\overline{\overline{\overline{T}} \cup T'} = \overline{\overline{T} \cup T'} = \overline{T'}$, *so* $p_0 \notin \overline{\overline{\overline{T}} \cup T'}$, contradicting (PR).

(2) Take $\{p_i : i \in \omega\}$ and put $m := m_{\bigwedge p_i}$, the model which makes all p_i true, in the top layer, all the other in the bottom layer. Let $m' \neq m$, $T' := \emptyset$, $T := Th(m, m')$. Then Then $\overline{\overline{T'}} = T'$, so $Con(\overline{\overline{T'}}, T)$, $\overline{\overline{T}} = Th(m')$, $\overline{\overline{\overline{T'}} \cup T} = T$.

□

We recall from [Sch04] the following Definition and part of the following Fact:

Definition 4.1

Let $\mathcal{Y} \subseteq \mathcal{P}(Z)$ be given and closed under arbitrary intersections.

(1) For $A \subseteq Z$, let $\widehat{A} := \bigcap\{X \in \mathcal{Y} : A \subseteq X\}$.

(2) For $B \in \mathcal{Y}$, we call $A \subseteq B$ a small subset of B iff there is no $X \in \mathcal{Y}$ s.t. $B - A \subseteq X \subset B$.

Intuitively, Z is the set of all models for $\mathcal{L}$, $\mathcal{Y}$ is $\boldsymbol{D}_{\mathcal{L}}$, and $\widehat{A} = M(Th(A))$, this is the intended application.

Fact 4.1

(1) If $\mathcal{Y} \subseteq \mathcal{P}(Z)$ is closed under arbitrary intersections and finite unions, $Z \in \mathcal{Y}$, $X, Y \subseteq Z$, then condition

$(\cup)$ $\widehat{X \cup Y} = \widehat{X} \cup \widehat{Y}$

holds, as do the following trivial ones:

(2) $X = Y \rightarrow \widehat{X} = \widehat{Y}$, but not conversely,

(3) $\widehat{X} \subseteq Y \rightarrow X \subseteq Y$, but not conversely,

(4) $X \subseteq \widehat{Y} \rightarrow \widehat{X} \subseteq \widehat{Y}$,

(4a) $\widehat{X \cap Y} \subseteq \widehat{X} \cap \widehat{Y}$.

In the intended application, the following hold:

(5) $Th(X) = Th(\widehat{X})$,

(6) $\widehat{A} \cap M(\psi) = \widehat{A \cap M(\psi)}$,

(7) $\widehat{A} - M(\phi) = \widehat{A - M(\phi)}$,

(8) $\widehat{A} - \widehat{B} \subseteq \widehat{A - B}$,

(9) Even if $A = \widehat{A}$, $B = \widehat{B}$, it is not necessarily true that $\widehat{A - B} \subseteq \widehat{A} - \widehat{B}$.

Proof:

(2), (3), (4), (5) are trivial.

(1) Let $\mathcal{Y}(U) := \{X \in \mathcal{Y} : U \subseteq X\}$. If $A \in \mathcal{Y}(X \cup Y)$, then $A \in \mathcal{Y}(X)$ and $A \in \mathcal{Y}(Y)$, so $\widehat{X \cup Y} \supseteq \widehat{X} \cup \widehat{Y}$. *If* $A \in \mathcal{Y}(X)$ *and* $B \in \mathcal{Y}(Y)$, *then* $A \cup B \in \mathcal{Y}(X \cup Y)$, *so* $\widehat{X \cup Y} \subseteq \widehat{X} \cup \widehat{Y} \in \mathcal{Y}$.

(4a) Let $X', Y' \in \mathcal{Y}$, $X \subseteq X'$, $Y \subseteq Y'$, then $X \cap Y \subseteq X' \cap Y'$, so $\widehat{X \cap Y} \subseteq \widehat{X} \cap \widehat{Y}$.

(6) $\widehat{A} \cap M(\psi) \supseteq \widehat{A \cap M(\psi)}$ by (4a). For "$\subseteq$": Let $A' \supseteq A \cap M(\psi)$, $A' \in \mathcal{Y}$, then $A' \cup M(\neg\psi) \in \mathcal{Y}$, $A' \cup M(\neg\psi) \supseteq A$, $(A' \cup M(\neg\psi)) \cap M(\psi) \subseteq A'$. So $\widehat{A} \cap M(\psi) \subseteq \widehat{A \cap M(\psi)}$.

(7) $X - M(\phi) = X \cap M(\neg\phi)$, and (6).

(8) Let $x \in \widehat{A} - \widehat{B}$, but $x \notin \widehat{A - B}$. So $x \models Th(A)$, and there is $\phi \in Th(B).x \not\models \phi$, and there is ψ s.t. $A - B \models \psi$, $x \not\models \psi$. By $B \subseteq M(\phi)$, $A \cap M(\neg\phi) \models \psi$, as $x \models Th(A)$, $x \models \neg\phi$, so $x \models \psi$, *contradiction*.

(9) Set $A := M_{\mathcal{L}}$, $B := \{m\}$ for $m \in M_{\mathcal{L}}$ arbitrary, $\mathcal{L}$ infinite. So $A = \widehat{A}$, $B = \widehat{B}$, but $\widehat{A - B} = A \neq A - B$.

□

4.2 Central condition (intersection)

We analyze the problem of (PR), seen in Example 4.1 (1) above, working in the intended application.

(PR) is equivalent to $M(\overline{\overline{T}} \cup T') \subseteq M(\overline{\overline{T \cup T'}})$. To show (PR) from (μPR), we argue as follows, the crucial point is marked by "?":

$M(\overline{\overline{T \cup T'}}) = M(Th(\mu(M_{T \cup T'}))) = \overbrace{\mu(M_{T \cup T'})} \supseteq \mu(M_{T \cup T'}) = \mu(M_T \cap M_{T'}) \supseteq$ (by (μPR)) $\mu(M_T) \cap M_{T'}$? $\overbrace{\mu(M_T)} \cap M_{T'} = M(Th(\mu(M_T))) \cap M_{T'} = M(\overline{\overline{T}}) \cap M_{T'} = M(\overline{\overline{T}} \cup T')$. If μ is definability preserving, then $\mu(M_T) = \overbrace{\mu(M_T)}$, so "?" above is equality, and everything is fine. In general, however, we have only $\mu(M_T) \subseteq \overbrace{\mu(M_T)}$, and the argument collapses.

But it is not necessary to impose $\mu(M_T) = \overbrace{\mu(M_T)}$, as we still have room to move: $\overbrace{\mu(M_{T \cup T'})} \supseteq \mu(M_{T \cup T'})$. (We do not consider here $\mu(M_T \cap M_{T'}) \supseteq \mu(M_T) \cap M_{T'}$ as room to move, as we are now interested only in questions related to definability preservation.) If we had $\overbrace{\mu(M_T)} \cap M_{T'} \subseteq \overbrace{\mu(M_T) \cap M_{T'}}$, we could use $\mu(M_T) \cap M_{T'} \subseteq \mu(M_T \cap M_{T'}) = \mu(M_{T \cup T'})$ and monotony of $\overbrace{.}$ to obtain $\overbrace{\mu(M_T)} \cap M_{T'} \subseteq \overbrace{\mu(M_T) \cap M_{T'}} \subseteq \overbrace{\mu(M_T \cap M_{T'})} = \overbrace{\mu(M_{T \cup T'})}$. If, for instance, $T' = \{\psi\}$, we have $\overbrace{\mu(M_T)} \cap M_{T'} = \overbrace{\mu(M_T) \cap M_{T'}}$ by Fact 4.1 (6). Thus, definability preservation is not the only solution to the problem.

We have seen above that $\overbrace{X \cup Y} = \overbrace{X} \cup \overbrace{Y}$, moreoever X-Y $= X \cap \boldsymbol{C}Y$ ($\boldsymbol{C}Y$ the set complement of Y), so, when considering boolean expressions of model sets (as we do in usual properties describing logics), the central question is

whether

$(\cap)$ $\overbrace{X \cap Y} = \overbrace{X} \cap \overbrace{Y}$.

We take a closer look at this question. "$\subseteq$" holds by Fact 4.1 (6). Using $(\cup)$ and monotony of $\overbrace{.}$, we have $\overbrace{X} \cap \overbrace{Y} = \overbrace{((X \cap Y) \cup (X - Y))} \cap \overbrace{((X \cap Y) \cup (Y - X))}$ $= (\overbrace{(X \cap Y)} \cup \overbrace{(X - Y)}) \cap (\overbrace{(X \cap Y)} \cup \overbrace{(Y - X)}) = \overbrace{X \cap Y} \cup (\overbrace{X \cap Y} \cap \overbrace{Y - X}) \cup (\overbrace{X \cap Y} \cap \overbrace{X - Y}) \cup (\overbrace{X - Y} \cap \overbrace{Y - X}) = \overbrace{X \cap Y} \cup (\overbrace{X - Y} \cap \overbrace{Y - X})$,

thus $\overbrace{X} \cap \overbrace{Y} \subseteq \overbrace{X \cap Y}$ iff

$(\cap')$ $\overbrace{Y - X} \cap \overbrace{X - Y} \subseteq \overbrace{X \cap Y}$ holds.

Intuitively speaking, the condition holds iff we cannot approximate any element both from X-Y and X-Y, which cannot be approximated from $X \cap Y$, too.

Note that in above Example 4.1 $X := \mu(M_T) = M_{\mathcal{L}} - \{n'\}$, $Y := M_{T'} = \{n, n'\}$, $\overbrace{X - Y} = M_{\mathcal{L}}$, $\overbrace{Y - X} = \{n'\}$, $\overbrace{X \cap Y} = \{n\}$, and $\overbrace{X} \cap \overbrace{Y} = \{n, n'\}$.

We consider now particular cases:

(1) In particular, if $X \cap Y = \emptyset$, by $\emptyset \in \mathcal{Y}$, $(\cap)$ holds iff $\overbrace{X} \cap \overbrace{Y} = \emptyset$.

(2) If $X \in \mathcal{Y}$ and $Y \in \mathcal{Y}$, then $\overbrace{X - Y} \subseteq X$ and $\overbrace{Y - X} \subseteq Y$, so $\overbrace{X - Y} \cap \overbrace{Y - X} \subseteq X \cap Y \subseteq \overbrace{X \cap Y}$ and $(\cap)$ trivially holds.

(3) $X \in \mathcal{Y}$ and $\boldsymbol{C}X \in \mathcal{Y}$ together also suffice - in these cases $\widehat{Y-X} \cap \widehat{X-Y} = \emptyset$: $\widehat{Y-X} = \widehat{Y \cap \boldsymbol{C}X} \subseteq \boldsymbol{C}X$, and $\widehat{X-Y} \subseteq X$, *so* $\widehat{Y-X} \cap \widehat{X-Y} \subseteq X \cap \boldsymbol{C}X = \emptyset \subseteq \widehat{X \cap Y}$. (The same holds, of course, for Y.) (In the intended application, such X will be $M(\phi)$ for some formula ϕ. But, a warning, $\mu(M(\phi))$ need not again be the $M(\psi)$ for some ψ.)

We turn to the properties of various structures and apply our results.

4.2.1 The minimal variant

We now take a look at other frequently used logical conditions. First, in the context on nonmonotonic logics, the following rules will always hold in smooth preferential structures, even if we consider full theories, and not necessarily definability preserving structures:

Fact 4.2

Also for full theories, and not necessarily definability preserving structures hold:

(1) (LLE), (RW), (AND), (REFLEX), by definition and ($\mu \subseteq$),

(2) (OR),

(3) (CM) in smooth structures,

(4) the infinitary version of (CUM) in smooth structures. In definability preserving structures, but also when considering only formulas hold:

(5) (PR),

(6) ($\mid\!\sim=$) in ranked structures.

Proof:

We use the corresponding algebraic properties.

(1) trivial.

(2) We have to show $T \mid\!\sim \phi$, $T' \mid\!\sim \phi \rightarrow T \vee T' \mid\!\sim \phi$: $\mu(T) \models \phi$, $\mu(T') \models \phi$. Thus $\mu(T \vee T') = \mu(M(T) \cup M(T')) \subseteq \mu(T) \cup \mu(T') \models \phi$.

(3) We have to show $T \mid\!\sim \beta$, $T \mid\!\sim \gamma \rightarrow T \cup \beta \mid\!\sim \gamma$: $\mu(T) \subseteq M(T \cup \beta) \subseteq M(T)$ (as $\mu(T) \subseteq M(T)$ and $\mu(T) \subseteq M(\beta)$), so $\mu(T) = \mu(T \cup \beta)$.

(4) Let $T \subseteq \overline{T'} \subseteq \overline{\overline{T}}$. Thus by (μCum) and $\mu(M_T) \subseteq M_{\overline{\overline{T}}} \subseteq M_{T'} \subseteq M_T$ $\mu(M_T) = \mu(M_{T'})$, so $\overline{\overline{T}} = Th(\mu(M_T)) = Th(\mu(M_{T'})) = \overline{\overline{T'}}$. (The proof given in [Sch04] uses definability preservation, but this is not necessary, as we see here.)

(5) See above discussion.

(6) $T \vdash T'$ iff $M(T) \subseteq M(T')$. $Con(\overline{\overline{\overline{T'}}}, T)$ iff $M(\overline{\overline{\overline{T'}}}) \cap M(T) \neq \emptyset$ iff $\overbrace{\mu(T')} \cap M(T) \neq \emptyset$ iff $\overbrace{\mu(T') \cap M(T)} \neq \emptyset$ iff $\mu(T') \cap M(T) \neq \emptyset$, if $\mu(T') \in \mathcal{Y}$ or $T = \phi$, as we saw above. So by rankedness $\mu(T) = \mu(T') \cap M(T) \rightarrow \overline{\overline{\overline{T}}} = Th(\mu(T)) = Th(\mu(T') \cap M(T))$, and $Th(\mu(T') \cap M(T)) = \overline{\overline{\overline{T'}}} \cup T$ if $T = \phi$ or $\mu(T') \in \mathcal{Y}$ again.

□

We turn to theory revision. The following definition and example, taken from [Sch04] shows, that the usual AGM axioms for theory revision fail in distance based structures in the general case, unless we require definability preservation.

Definition 4.2

We summarize the AGM postulates $(K * 7)$ and $(K * 8)$ in $(*4)$:

$(*4)$ If $T * T'$ is consistent with T'', then $T * (T' \cup T'') = \overline{(T * T') \cup T''}$.

Example 4.2

Consider an infinite propositional language $\mathcal{L}$.

Let X be an infinite set of models, m, m_1, m_2 be models for $\mathcal{L}$. Arrange the models of $\mathcal{L}$ in the real plane s.t. all $x \in X$ have the same distance < 2 (in the real plane) from m, m_2 has distance 2 from m, and m_1 has distance 3 from m.

Let T, T_1, T_2 be complete (consistent) theories, T' a theory with infinitely many models, $M(T) = \{m\}$, $M(T_1) = \{m_1\}$, $M(T_2) = \{m_2\}$. $M(T') = X \cup \{m_1, m_2\}$, $M(T'') = \{m_1, m_2\}$.

Assume $Th(X) = T'$, so X will not be definable by a theory.

Then $M(T) \mid M(T') = X$, but $T * T' = Th(X) = T'$. So $T * T'$ is consistent with T'', and $\overline{(T * T') \cup T''} = T''$. But $T' \cup T'' = T''$, and $T * (T' \cup T'') = T_2 \neq T''$, contradicting $(*4)$.

□

We show now that the version with formulas only holds here, too, just as does above (PR), when we consider formulas only - this is needed below for T'' only. This was already shown in [Sch04], we give now a proof based on our new principles.

Fact 4.3

$(*4)$ holds when considering only formulas.

Proof:

When we fix the left hand side, the structure is ranked, so $Con(T * T', T'')$ implies $(M_T \mid M_{T'}) \cap M_{T''} \neq \emptyset$ by $T'' = \{\psi\}$ and thus $M_T \mid M_{T' \cup T''} = M_T \mid (M_{T'} \cap M_{T''}) = (M_T \mid M_{T'}) \cap M_{T''}$. So $M(T * (T' \cup T'')) = \overbrace{M_T \mid M_{T' \cup T''}} = \overbrace{(M_T \mid M_{T'}) \cap M_{T''}} =$ (by $T'' = \{\psi\}$, see above) $\overbrace{(M_T \mid M_{T'})} \cap \overbrace{M_{T''}} = \overbrace{(M_T \mid M_{T'})} \cap M_{T''} = M((T * T') \cup T'')$, and $T * (T' \cup T'') = \overline{(T * T') \cup T''}$. □

4.2.2 The limit variant

We begin with some simple logical facts about the limit version.

We abbreviate $\Lambda(T) := \Lambda(M(T))$ etc.

Fact 4.4

(1) $A \in \Lambda(T) \to M(\overline{\overline{T}}) \subseteq \overbrace{A}$

(2) $M(\overline{\overline{T}}) = \bigcap\{\overbrace{A} : A \in \Lambda(T)\}$

(2a) $M(\overline{\overline{T'}}) \models \sigma \to \exists B \in \Lambda(T').\ \overbrace{B} \models \sigma$

(3) $M(\overline{\overline{T'}}) \cap M(T) \models \sigma \to \exists B \in \Lambda(T').\ \overbrace{B} \cap M(T) \models \sigma$.

Proof:

(1) Let $M(\overline{\overline{T}}) \not\subseteq \overbrace{A}$, so there is ϕ, $\overbrace{A} \models \phi$, so $A \models \phi$, but $M(\overline{\overline{T}}) \not\models \phi$, so $T \not\mid\!\sim \phi$, *contradiction*.

(2) "$\subseteq$" by (1). Let $x \in \bigcap\{\overbrace{A} : A \in \Lambda(T)\} \to \forall A \in \Lambda(T).x \models Th(A) \to x \models \overline{\overline{T}}$.

(2a) $M(\overline{\overline{T'}}) \models \sigma \to T' \mid\!\sim \sigma \to \exists B \in \Lambda(T').B \models \sigma$. But $B \models \sigma \to \overbrace{B} \models \sigma$.

(3) $M(\overline{\overline{T'}}) \cap M(T) \models \sigma \to \overline{\overline{T'}} \cup T \vdash \sigma \to \exists \tau_1 \ldots \tau_n \in \overline{\overline{T'}}$ s.t. $T \cup \{\tau_1, \ldots, \tau_n\} \vdash \sigma$, so $\exists B \in \Lambda(T').Th(B) \cup T \vdash \sigma$. So $M(Th(B)) \cap M(T) \models \sigma \to \overbrace{B} \cap M(T) \models \sigma$.

□

We saw in Example 3.2 and its discussion the problems which might arise in the limit version, even if the algebraic behaviour is correct.

This analysis leads us to consider the following facts:

Fact 4.5

(1) Let $\forall B \in \Lambda(T') \exists A \in \Lambda(T).A \subseteq B \cap M(T)$, then $\overline{\overline{\overline{T'}} \cup T} \subseteq \overline{\overline{T}}$.

Let, in addition, $\{B \in \Lambda(T') : \overbrace{B} \cap \overbrace{M(T)} = \overbrace{B \cap M(T)}\}$ be cofinal in $\Lambda(T')$. Then

(2) $Con(\overline{\overline{T'}}, T)$ implies $\forall A \in \Lambda(T').A \cap M(T) \neq \emptyset$.

(3) $\forall A \in \Lambda(T) \exists B \in \Lambda(T').B \cap M(T) \subseteq A$ implies $\overline{\overline{T}} \subseteq \overline{\overline{\overline{T'}} \cup T}$.

Note that $M(T) = \overbrace{M(T)}$, so we could also have written $\overbrace{B} \cap M(T) = \overbrace{B \cap M(T)}$, but above way of writing stresses more the essential condition $\overbrace{X} \cap \overbrace{Y} = \overbrace{X \cap Y}$.

Proof:

(1) Let $\overline{\overline{T'}} \cup T \vdash \sigma$, so $\exists B \in \Lambda(T').\overbrace{B} \cap M(T) \models \sigma$ by Fact 4.4, (3) above (using compactness). Thus $\exists A \in \Lambda(T).A \subseteq B' \cap M(T) \models \sigma$ by prerequisite, so $\sigma \in \overline{\overline{T}}$.

(2) Let $Con(\overline{\overline{T'}}, T)$, so $M(\overline{\overline{T'}}) \cap M(T) \neq \emptyset$. $M(\overline{\overline{T'}}) = \bigcap\{\overbrace{A} : A \in \Lambda(T')\}$ by Fact 4.4 (2), so $\forall A \in \Lambda(T').\overbrace{A} \cap M(T) \neq \emptyset$. As cofinally often $\overbrace{A} \cap M(T) = \overbrace{A \cap M(T)}$, $\forall A \in \Lambda(T').\overbrace{A \cap M(T)} \neq \emptyset$, so $\forall A \in \Lambda(T').A \cap M(T) \neq \emptyset$ by $\overbrace{\emptyset} = \emptyset$.

(3) Let $\sigma \in \overline{\overline{T}}$, so $T \mid\!\sim \sigma$, so $\exists A \in \Lambda(T).A \models \sigma$, so $\exists B \in \Lambda(T').B \cap M(T) \subseteq A$ by prerequisite, so $\exists B \in \Lambda(T').B \cap M(T) \subseteq A \wedge \overbrace{B} \cap \overbrace{M(T)} = \overbrace{B \cap M(T)}$. So for such B $\overbrace{B} \cap \overbrace{M(T)} = \overbrace{B \cap M(T)} \subseteq \overbrace{A} \models \sigma$. By Fact 4.4 (1) $M(\overline{\overline{T'}}) \subseteq \overbrace{B}$, so $M(\overline{\overline{T'}}) \cap M(T) \models \sigma$, so $\overline{\overline{T'}} \cup T \vdash \sigma$.

□

We obtain now as easy corollaries of a more general situation the following properties shown in [Sch04] by direct proofs. Thus, we have the trivialization results shown there.

Corollary 4.6

Let the structure be transitive.

(1) Let $\{B \in \Lambda(T') : \overbrace{B} \cap \overbrace{M(T)} = \overbrace{B \cap M(T)}\}$ be cofinal in $\Lambda(T')$, then:

(PR) $T \vdash T' \rightarrow \overline{\overline{T}} \subseteq \overline{\overline{\overline{T'}} \cup T}$.

(2) $\overline{\overline{\phi \wedge \phi'}} \subseteq \overline{\overline{\overline{\phi}} \cup \{\phi'\}}$

If the structure is ranked, then also:

(3) Let $\{B \in \Lambda(T') : \overbrace{B} \cap \overbrace{M(T)} = \overbrace{B \cap M(T)}\}$ be cofinal in $\Lambda(T')$, then:

($\mid\!\sim=$) $T \vdash T'$, $Con(\overline{\overline{\overline{T'}}}, T) \rightarrow \overline{\overline{\overline{T}}} = \overline{\overline{\overline{\overline{T'}}} \cup T}$

(4) $T \mid\!\not\sim \neg\gamma \rightarrow \overline{\overline{\overline{T}}} = \overline{\overline{\overline{T \cup \{\gamma\}}}}$

Proof:

(1) $\forall A \in \Lambda(M(T)) \exists B \in \Lambda(M(T')). B \cap M(T) \subseteq A$ by Fact 3.3 (2). So the result follows from Fact 4.5 (3).

(2) Set $T' := \{\phi\}$, $T := \{\phi, \phi'\}$. Then for $B \in \Lambda(T')$ $\overbrace{B} \cap M(T) = \overbrace{B} \cap M(\phi') = \overbrace{B \cap M(\phi')}$ by Fact 4.1 (6), so the result follows by (1).

(3) Let $Con(\overline{\overline{\overline{T'}}}, T)$, then by Fact 4.5 (2) $\forall A \in \Lambda(T'). A \cap M(T) \neq \emptyset$, so by Fact 3.3 (4) $\forall B \in \Lambda(T') \exists A \in \Lambda(T). A \subseteq B \cap M(T)$, so $\overline{\overline{\overline{\overline{T'}}} \cup T} \subseteq \overline{\overline{\overline{T}}}$ by Fact 4.3 (1). The other direction follows from (1).

(4) Set $T := T' \cup \{\gamma\}$. Then for $B \in \Lambda(T')$ $\overbrace{B} \cap M(T) = \overbrace{B} \cap M(\gamma) = \overbrace{B \cap M(\gamma)}$ again by Fact 4.1 (6), so the result follows from (3).

□

4.3 A simplification of [Sch04]

Note that in Sections 3.2 and 3.3 of [Sch04], as well as in Proposition 4.2.2 of [Sch04] we have characterized $\mu : \mathcal{Y} \rightarrow \mathcal{Y}$ or $| : \mathcal{Y} \times \mathcal{Y} \rightarrow \mathcal{Y}$, but a closer inspection of the proofs shows that the destination can as well be assumed $\mathcal{P}(Z)$, consequently we can simply re-use above algebraic representation results also for the not definability preserving case. (Note that the easy direction of all these results work for destination $\mathcal{P}(Z)$, too.) In particular, also the proof for the not definability preserving case of revision in [Sch04] can be simplified - but we will not go into details here.

$(\cup)$ and $(\bigcap)$ are assumed to hold now - we need $(\bigcap)$ for $\overbrace{\cdot}$.

The central functions and conditions to consider are summarized in the following definition.

Definition 4.3

Let $\mu : \mathcal{Y} \rightarrow \mathcal{Y}$, we define $\mu_i : \mathcal{Y} \rightarrow \mathcal{P}(Z)$:

$\mu_0(U) := \{x \in U : \neg\exists Y \in \mathcal{Y}(Y \subseteq U \text{ and } x \in Y - \mu(Y))\}$,

$\mu_1(U) := \{x \in U : \neg\exists Y \in \mathcal{Y}(\mu(Y) \subseteq U \text{ and } x \in Y - \mu(Y))\}$,

$\mu_2(U) := \{x \in U : \neg\exists Y \in \mathcal{Y}(\mu(U \cup Y) \subseteq U \text{ and } x \in Y - \mu(Y))\}$

(note that we use $(\cup)$ here),

$\mu_3(U) := \{x \in U : \forall y \in U.x \in \mu(\{x, y\})\}$

(we use here $(\cup)$ and that singletons are in $\mathcal{Y}$)

$(\mu PR0)$ $\mu(U) - \mu_0(U)$ is small,

$(\mu PR1)$ $\mu(U) - \mu_1(U)$ is small,

$(\mu PR2)$ $\mu(U) - \mu_2(U)$ is small,

$(\mu PR3)$ $\mu(U) - \mu_3(U)$ is small.

$(\mu PR0)$ with its function will be the one to consider for general preferential structures, $(\mu PR2)$ the one for smooth structures. Unfortunately, we cannot use $(\mu PR0)$ in the smooth case, too, as Example 4.3 below will show. This sheds some doubt on the possibility to find an easy common approach to all cases of not definability preserving preferential, and perhaps other, structures. The next best guess, $(\mu PR1)$ will not work either, as the same example shows - or by Fact 4.7 (10), if μ satisfies (μCum), then $\mu_0(U) = \mu_1(U)$. $(\mu PR3)$ and μ_3 are used for ranked structures.

Note that in our context, μ will not necessarily respect (μPR). Thus, if e.g. $x \in Y - \mu(Y)$, and $\mu(Y) \subseteq U$, we cannot necessarily conclude that $x \notin \mu(U \cup Y)$ - the fact that x is minimized in $U \cup Y$ might be hidden by the bigger $\mu(U \cup Y)$.

The strategy of representation without definability preservation will in all cases be very simple: Under sufficient conditions, among them smallness (μPRi) as described above, the corresponding function μ_i has all the properties to guarantee representation by a corresponding structures, and we can just take our representation theorems for the dp case, to show this. Using smallness again, we can show that we have obtained a sufficient approximation - see Propositions 4.9, 4.10, and 4.15.

We first show some properties for the μ_i, $i = 0, 1, 2$. A corresponding result for μ_3 is given in Fact 4.13 below. (The conditions and results are sufficiently different for μ_3 to make a separation more natural.)

Property (9) of the following Fact 4.7 fails for μ_0 and μ_1, as Example 4.3 below will show. We will therefore work in the smooth case with μ_2.

4.3.1 The general and the smooth case

Fact 4.7

(This is partly Fact 5.2.6 in [Sch04].)

Recall that $\mathcal{Y}$ is closed under $(\cup)$, and $\mu : \mathcal{Y} \to \mathcal{Y}$. Let A, B, U, U', X, Y be elements of $\mathcal{Y}$ and the μ_i be defined from μ as in Definition 4.3. i will here be 0,1, or 2, but not 3.

(1) Let μ satisfy $(\mu \subseteq)$, then $\mu_1(X) \subseteq \mu_0(X)$ and $\mu_3(X) \subseteq \mu_0(X)$,

(2) Let μ satisfy $(\mu \subseteq)$ and (μCum), then $\mu(U \cup U') \subseteq U \leftrightarrow \mu(U \cup U') = \mu(U)$,

(3) Let μ satisfy $(\mu \subseteq)$, then $\mu_i(U) \subseteq \mu(U)$, and $\mu_i(U) \subseteq U$,

(4) Let μ satisfy $(\mu \subseteq)$ and one of the (μPRi), then $\mu(A \cup B) \subseteq \mu(A) \cup \mu(B)$,

(5) Let μ satisfy $(\mu \subseteq)$ and one of the (μPRi), then $\mu_2(X) \subseteq \mu_1(X)$,

(6) Let μ satisfy $(\mu \subseteq)$, (μPRi), then $\mu_i(U) \subseteq U' \leftrightarrow \mu(U) \subseteq U'$,

(7) Let μ satisfy $(\mu \subseteq)$ and one of the (μPRi), then $X \subseteq Y$, $\mu(X \cup U) \subseteq X \to \mu(Y \cup U) \subseteq Y$,

(8) Let μ satisfy $(\mu \subseteq)$ and one of the (μPRi), then $X \subseteq Y \to X \cap \mu_i(Y) \subseteq \mu_i(X)$ - (μPR) for μ_i, (more precisely, only for μ_2 we need the prerequisites, in the other cases the definition suffices)

(9) Let μ satisfy $(\mu \subseteq)$, $(\mu PR2)$, (μCum), then $\mu_2(X) \subseteq Y \subseteq X \to \mu_2(X) = \mu_2(Y)$ - (μCum) for μ_2.

(10) $(\mu \subseteq)$ and (μCum) for μ entail $\mu_0(U) = \mu_1(U)$.

Proof:

(1) $\mu_1(X) \subseteq \mu_0(X)$ follows from $(\mu \subseteq)$ for μ. For μ_2 : By $Y \subseteq U$, $U \cup Y = U$, so $\mu(U) \subseteq U$ by $(\mu \subseteq)$.

(2) $\mu(U \cup U') \subseteq U \subseteq U \cup U' \to_{(\mu CUM)} \mu(U \cup U') = \mu(U)$.

(3) $\mu_i(U) \subseteq U$ by definition. To show $\mu_i(U) \subseteq \mu(U)$, take in all three cases $Y := U$, and use for $i = 1, 2$ $(\mu \subseteq)$.

(4) By definition of μ_0, we have $\mu_0(A \cup B) \subseteq A \cup B$, $\mu_0(A \cup B) \cap (A - \mu(A)) = \emptyset$, $\mu_0(A \cup B) \cap (B - \mu(B)) = \emptyset$, so $\mu_0(A \cup B) \cap A \subseteq \mu(A)$, $\mu_0(A \cup B) \cap B \subseteq \mu(B)$, and $\mu_0(A \cup B) \subseteq \mu(A) \cup \mu(B)$. By $\mu : \mathcal{Y} \to \mathcal{Y}$ and $(\cup)$, $\mu(A) \cup \mu(B) \in \mathcal{Y}$. Moreover, by (3) $\mu_0(A \cup B) \subseteq \mu(A \cup B)$, so $\mu_0(A \cup B) \subseteq (\mu(A) \cup \mu(B)) \cap \mu(A \cup B)$, so by (0) $\mu_i(A \cup B) \subseteq (\mu(A) \cup \mu(B)) \cap \mu(A \cup B)$. If $\mu(A \cup B) \not\subseteq \mu(A) \cup \mu(B)$, then $(\mu(A) \cup \mu(B)) \cap \mu(A \cup B) \subset \mu(A \cup B)$, contradicting (μPRi).

(5) Let $Y \in \mathcal{Y}$, $\mu(Y) \subseteq U$, $x \in Y - \mu(Y)$, then (by (4)) $\mu(U \cup Y) \subseteq \mu(U) \cup \mu(Y) \subseteq U$.

(6) "$\leftarrow$" by (3). "$\to$": By (μPRi), $\mu(U) - \mu_i(U)$ is small, so there is no $X \in \mathcal{Y}$ s.t. $\mu_i(U) \subseteq X \subset \mu(U)$. If there were $U' \in \mathcal{Y}$ s.t. $\mu_i(U) \subseteq U'$, but $\mu(U) \not\subseteq U'$, then for $X := U' \cap \mu(U) \in \mathcal{Y}$, $\mu_i(U) \subseteq X \subset \mu(U)$, *contradiction.*

(7) $\mu(Y \cup U) = \mu(Y \cup X \cup U) \subseteq_{(4)} \mu(Y) \cup \mu(X \cup U) \subseteq Y \cup X = Y$.

(8) For $i = 0, 1$: Let $x \in X - \mu_0(X)$, then there is A s.t. $A \subseteq X$, $x \in A - \mu(A)$, so

$A \subseteq Y$. The case $i = 1$ is similar. We need here only the definitions. For $i = 2$: Let $x \in X - \mu_2(X)$, A s.t. $x \in A - \mu(A)$, $\mu(X \cup A) \subseteq X$, then by (7) $\mu(Y \cup A) \subseteq Y$.

(9) "$\subseteq''$: Let $x \in \mu_2(X)$, so $x \in Y$, and $x \in \mu_2(Y)$ by (8). "$\supseteq''$: Let $x \in \mu_2(Y)$, so $x \in X$. Suppose $x \notin \mu_2(X)$, so there is $U \in \mathcal{Y}$ s.t. $x \in U - \mu(U)$ and $\mu(X \cup U) \subseteq X$. Note that by $\mu(X \cup U) \subseteq X$ and (2), $\mu(X \cup U) = \mu(X)$. Now, $\mu_2(X) \subseteq Y$, so by (6) $\mu(X) \subseteq Y$, thus $\mu(X \cup U) = \mu(X) \subseteq Y \subseteq Y \cup U \subseteq X \cup U$, so $\mu(Y \cup U) = \mu(X \cup U) = \mu(X) \subseteq Y$ by (μCum), so $x \notin \mu_2(Y)$, *contradiction*.

(10) $\mu_1(U) \subseteq \mu_0(U)$ by (1). Let Y s.t. $\mu(Y) \subseteq U$, $x \in Y - \mu(Y)$, $x \in U$. Consider $Y \cap U$, $x \in Y \cap U$, $\mu(Y) \subseteq Y \cap U \subseteq Y$, so $\mu(Y) = \mu(Y \cap U)$ by (μCum), and $x \notin \mu(Y \cap U)$. Thus, $\mu_0(U) \subseteq \mu_1(U)$.

□

Fact 4.8

$(\mu PR0)$, $(\mu PR1)$, $(\mu PR2)$ are equivalent in the presence of $(\mu \subseteq)$, (μCum) for μ.

(Recall that $(\cup)$ and $(\cap)$ are assumed to hold.)

Proof:

We first show $(\mu PR1) \leftrightarrow (\mu PR2)$.

(1) Suppose $(\mu PR2)$ holds. By $(\mu PR2)$ and (5), $\mu_2(U) \subseteq \mu_1(U)$, so $\mu(U) - \mu_1(U) \subseteq \mu(U) - \mu_2(U)$. By $(\mu PR2)$, $\mu(U) - \mu_2(U)$ is small, then so is $\mu(U) - \mu_1(U)$, so $(\mu PR1)$ holds.

(2) Suppose $(\mu PR1)$ holds, and $(\mu PR2)$ fails. By failure of $(\mu PR2)$, there is $X \in \mathcal{Y}$ s.t. $\mu_2(U) \subseteq X \subset \mu(U)$. Let $x \in \mu(U)$–X, as $x \notin \mu_2(U)$, there is Y s.t. $\mu(U \cup Y) \subseteq U$, $x \in Y - \mu(Y)$. Let $Z := Y \cup X$. By Fact 4.7 (4) $\mu(Y \cup X) \subseteq \mu(Y) \cup \mu(X)$, so $x \notin \mu(Y \cup X)$. Moreover, $\mu(U \cup X \cup Y) \subseteq \mu(U \cup Y) \cup \mu(X)$ by Fact 4.7 (4), $\mu(U \cup Y) \subseteq U$, $\mu(X) \subseteq X \subseteq \mu(U) \subseteq U$ by prerequisite, so $\mu(U \cup X \cup Y) \subseteq U \subseteq U \cup Y \subseteq U \cup X \cup Y$, so $\mu(U \cup X \cup Y) = \mu(U \cup Y)$. Thus $\mu(Y \cup U \cup X) = \mu(Y \cup U) \subseteq Y \cup X \subseteq Y \cup U \cup X$, so $\mu(Y \cup X) = \mu(Y \cup U \cup X) = \mu(Y \cup U) \subseteq U$. Thus, $x \notin \mu_1(U)$, and $\mu_1(U) \subseteq X$, too, a contradiction.

(3) Finally, by Fact 4.7, (10), $\mu_0(U) = \mu_1(U)$ if (μCum) holds for μ.

□

Here is an example which shows that Fact 4.7, (9) may fail for μ_0 and μ_1.

Example 4.3

Consider $\mathcal{L}$ with $v(\mathcal{L}) := \{p_i : i \in \omega\}$. Let $m \not\models p_0$, let $m' \in M(p_0)$ arbitrary. Make for each $n \in M(p_0) - \{m'\}$ one copy of m, likewise of m', set $< m, n > \prec < m', n >$ for all n, and $n \prec < m, n >$, $n \prec < m', n >$ for all n. The resulting structure $\mathcal{Z}$ is smooth and transitive. Let $\mathcal{Y} := \boldsymbol{D}_{\mathcal{L}}$, define $\mu(X) := \overbrace{\mu_{\mathcal{Z}}(X)}$ for $X \in \mathcal{Y}$.

Let $m' \in X - \mu_{\mathcal{Z}}(X)$. Then $m \in X$, or $M(p_0) \subseteq X$. In the latter case, as all m'' s.t. $m'' \neq m'$, $m'' \models p_0$ are minimal, $M(p_0) - \{m'\} \subseteq \mu_{\mathcal{Z}}(X)$, so $m' \in \overbrace{\mu_{\mathcal{Z}}(X)} = \mu(X)$. Thus, as $\mu_{\mathcal{Z}}(X) \subseteq \mu(X)$, if $m' \in X - \mu(X)$, then $m \in X$.

Define now $X := M(p_0) \cup \{m\}$, $Y := M(p_0)$.

We first show that μ_0 does not satisfy (μCum). $\mu_0(X) := \{x \in X : \neg \exists A \in \mathcal{Y}(A \subseteq X : x \in A - \mu(A))\}$. $m \notin \mu_0(X)$, as $m \notin \mu(X) = \overbrace{\mu_{\mathcal{Z}}(X)}$. Moreover, $m' \notin \mu_0(X)$, as $\{m, m'\} \in \mathcal{Y}$, $\{m, m'\} \subseteq X$, and $\mu(\{m, m'\}) = \mu_{\mathcal{Z}}(\{m, m'\}) = \{m\}$. So $\mu_0(X) \subseteq Y \subseteq X$. Consider now $\mu_0(Y)$. As $m \notin Y$, for any $A \in \mathcal{Y}$, $A \subseteq Y$, if $m' \in A$, then $m' \in \mu(A)$, too, by above argument, so $m' \in \mu_0(Y)$, and μ_0 does not satisfy (μCum).

We turn to μ_1.

By Fact 4.7 (1), $\mu_1(X) \subseteq \mu_0(X)$, so $m, m' \notin \mu_1(X)$, and again $\mu_1(X) \subseteq Y \subseteq X$. Consider again $\mu_1(Y)$. As $m \notin Y$, for any $A \in \mathcal{Y}$, $\mu(A) \subseteq Y$, if $m' \in A$, then $m' \in \mu(A)$, too: if $M(p_0) - \{m'\} \subseteq A$, then $m' \in \overbrace{\mu_{\mathcal{Z}}(A)}$, if $M(p_0) - \{m'\} \not\subseteq A$, but $m' \in A$, then either $m' \in \mu_{\mathcal{Z}}(A)$, or $m \in \mu_{\mathcal{Z}}(A) \subseteq \mu(A)$, but $m \notin Y$. Thus, (μCum) fails for μ_1, too.

It remains to show that μ satisfies $(\mu \subseteq)$, (μCum), $(\mu PR0)$, $(\mu PR1)$. Note that by Fact .... $\mu_{\mathcal{Z}}$ satisfies (μCum), as $\mathcal{Z}$ is smooth. $(\mu \subseteq)$ is trivial. We show (μPRi) for $i = 0, 1$. As $\mu_{\mathcal{Z}}(A) \subseteq \mu(A)$, by (μPR) and (μCum) for $\mu_{\mathcal{Z}}$, $\mu_{\mathcal{Z}}(X) \subseteq \mu_0(X)$ and $\mu_{\mathcal{Z}}(X) \subseteq \mu_1(X)$.

To see this: $\mu_{\mathcal{Z}}(X) \subseteq \mu_0(X)$: Let $x \in X - \mu_0(X)$, then there is Y s.t. $x \in Y - \mu(Y)$.$Y \subseteq X$, but $\mu_{\mathcal{Z}}(Y) \subseteq \mu(Y)$, so by $Y \subseteq X$ and (μPR) for $\mu_{\mathcal{Z}}$ $x \notin \mu_{\mathcal{Z}}(X)$. $\mu_{\mathcal{Z}}(X) \subseteq \mu_1(X)$: Let $x \in X - \mu_1(X)$, then there is Y s.t. $x \in Y - \mu(Y)$, $\mu(Y) \subseteq X$, so $x \in Y - \mu_{\mathcal{Z}}(Y)$ and $\mu_{\mathcal{Z}}(Y) \subseteq X$. $\mu_{\mathcal{Z}}(X \cup Y) \subseteq \mu_{\mathcal{Z}}(X) \cup \mu_{\mathcal{Z}}(Y) \subseteq X \subseteq X \cup Y$, so $\mu_{\mathcal{Z}}(X \cup Y) = \mu_{\mathcal{Z}}(X)$ by (μCum) for $\mu_{\mathcal{Z}}$. $x \in Y - \mu_{\mathcal{Z}}(Y) \rightarrow x \notin \mu_{\mathcal{Z}}(X \cup Y)$ by (μPR) for $\mu_{\mathcal{Z}}$, so $x \notin \mu_{\mathcal{Z}}(X)$.

But by Fact 4.7, (3) $\mu_i(X) \subseteq \mu(X)$. As by definition, $\mu(X) - \mu_{\mathcal{Z}}(X)$ is small, (μPRi) hold for $i = 0, 1$. It remains to show (μCum) for μ. Let $\mu(X) \subseteq Y \subseteq X$, then $\mu_{\mathcal{Z}}(X) \subseteq \mu(X) \subseteq Y \subseteq X$, so by (μCum) for $\mu_{\mathcal{Z}}$ $\mu_{\mathcal{Z}}(X) = \mu_{\mathcal{Z}}(Y)$, so by definition of μ, $\mu(X) = \mu(Y)$.

(Note that by Fact 4.7 (10), $\mu_0 = \mu_1$ follows from (μCum) for μ, so we could have demonstrated part of the properties also differently.)

□

By Fact 4.7 (3) and (8) and Proposition 3.2.4 in [Sch04], μ_0 has a representation by a (transitive) preferential structure, if $\mu : \mathcal{Y} \to \mathcal{Y}$ satisfies $(\mu \subseteq)$ and $(\mu PR0)$, and μ_0 is defined as in Definition 4.3.

We thus have (taken from [Sch04]):

Proposition 4.9

Let Z be an arbitrary set, $\mathcal{Y} \subseteq \mathcal{P}(Z)$, $\mu : \mathcal{Y} \to \mathcal{Y}$, $\mathcal{Y}$ closed under arbitrary intersections and finite unions, and $\emptyset, Z \in \mathcal{Y}$, and let $\overbrace{\cdot}$ be defined wrt. $\mathcal{Y}$.

(a) If μ satisfies $(\mu \subseteq)$, $(\mu PR0)$, then there is a transitive preferential structure $\mathcal{Z}$ over Z s.t. for all $U \in \mathcal{Y}$ $\mu(U) = \overbrace{\mu_{\mathcal{Z}}(U)}$.

(b) If $\mathcal{Z}$ is a preferential structure over Z and $\mu : \mathcal{Y} \to \mathcal{Y}$ s.t. for all $U \in \mathcal{Y}$ $\mu(U) = \overbrace{\mu_{\mathcal{Z}}(U)}$, then μ satisfies $(\mu \subseteq)$, $(\mu PR0)$.

Proof:

(a) Let μ satisfy $(\mu \subseteq)$, $(\mu PR0)$. μ_0 as defined in Definition 4.3 satisfies properties $(\mu \subseteq)$, (μPR) by Fact 4.7, (3) and (8). Thus, by Proposition 3.2.4 in [Sch04], there is a transitive structure $\mathcal{Z}$ over Z s.t. $\mu_0 = \mu_{\mathcal{Z}}$, but by $(\mu PR0)$ $\mu(U) = \overbrace{\mu_0(U)} = \overbrace{\mu_{\mathcal{Z}}(U)}$ for $U \in \mathcal{Y}$.

(b) $(\mu \subseteq) : \mu_{\mathcal{Z}}(U) \subseteq U$, so by $U \in \mathcal{Y}$ $\mu(U) = \overbrace{\mu_{\mathcal{Z}}(U)} \subseteq U$.

$(\mu PR0)$: If $(\mu PR0)$ is false, there is $U \in \mathcal{Y}$ s.t. for $U' := \bigcup\{Y' - \mu(Y') : Y' \in \mathcal{Y}, Y' \subseteq U\}$ $\overbrace{\mu(U) - U'} \subset \mu(U)$. By $\mu_{\mathcal{Z}}(Y') \subseteq \mu(Y')$, $Y' - \mu(Y') \subseteq Y' - \mu_{\mathcal{Z}}(Y')$. No copy of any $x \in Y' - \mu_{\mathcal{Z}}(Y')$ with $Y' \subseteq U$, $Y' \in \mathcal{Y}$ can be minimal in $\mathcal{Z} \lceil U$. Thus, by $\mu_{\mathcal{Z}}(U) \subseteq \mu(U)$, $\mu_{\mathcal{Z}}(U) \subseteq \mu(U) - U'$, so $\overbrace{\mu_{\mathcal{Z}}(U)} \subseteq \overbrace{\mu(U) - U'} \subset \mu(U)$, *contradiction.*

□

We turn to the smooth case.

If $\mu : \mathcal{Y} \to \mathcal{Y}$ satisfies $(\mu \subseteq)$, $(\mu PR2)$, (μCUM) and μ_2 is defined from μ as in Definition 4.3, then μ_2 satisfies $(\mu \subseteq)$, (μPR), (μCum) by Fact 4.7 (3), (8), and (9), and can thus be represented by a (transitive) smooth structure, by Proposition 3.3.8 in [Sch04], and we finally have (taken from [Sch04]):

Proposition 4.10

Let Z be an arbitrary set, $\mathcal{Y} \subseteq \mathcal{P}(Z)$, $\mu : \mathcal{Y} \to \mathcal{Y}$, $\mathcal{Y}$ closed under arbitrary intersections and finite unions, and $\emptyset, Z \in \mathcal{Y}$, and let $\overbrace{\cdot}$ be defined wrt. $\mathcal{Y}$.

(a) If μ satisfies $(\mu \subseteq)$, $(\mu PR2)$, (μCUM), then there is a transitive smooth preferential structure $\mathcal{Z}$ over Z s.t. for all $U \in \mathcal{Y}$ $\mu(U) = \overbrace{\mu_{\mathcal{Z}}(U)}$.
(b) If $\mathcal{Z}$ is a smooth preferential structure over Z and $\mu : \mathcal{Y} \to \mathcal{Y}$ s.t. for all $U \in \mathcal{Y}$ $\mu(U) = \overbrace{\mu_{\mathcal{Z}}(U)}$, then μ satisfies $(\mu \subseteq)$, $(\mu PR2)$, (μCUM).

Proof:

(a) If μ satisfies $(\mu \subseteq)$, $(\mu PR2)$, (μCUM), then μ_2 defined from μ as in Definition 4.3 satisfies $(\mu \subseteq)$, (μPR), (μCUM) by Fact 4.7 (3), (8) and (9). Thus, by Proposition 3.3.8 in [Sch04], there is a smooth transitive preferential structure $\mathcal{Z}$ over Z s.t. $\mu_2 = \mu_{\mathcal{Z}}$, but by $(\mu PR2)$ $\mu(U) = \overbrace{\mu_2(U)} = \overbrace{\mu_{\mathcal{Z}}(U)}$.

(b) $(\mu \subseteq) : \mu_{\mathcal{Z}}(U) \subseteq U \to \mu(U) = \overbrace{\mu_{\mathcal{Z}}(U)} \subseteq U$ by $U \in \mathcal{Y}$.

$(\mu PR2)$: If $(\mu PR2)$ fails, then there is $U \in \mathcal{Y}$ s.t. for $U' := \bigcup\{Y' - \mu(Y') : Y' \in \mathcal{Y}, \mu(U \cup Y') \subseteq U\}$ $\overbrace{\mu(U) - U'} \subset \mu(U)$.

By $\mu_{\mathcal{Z}}(Y') \subseteq \mu(Y')$, $Y' - \mu(Y') \subseteq Y' - \mu_{\mathcal{Z}}(Y')$. But no copy of any $x \in Y' - \mu_{\mathcal{Z}}(Y')$ with $\mu_{\mathcal{Z}}(U \cup Y') \subseteq \mu(U \cup Y') \subseteq U$ can be minimal in $\mathcal{Z}\lceil U$: As $x \in Y' - \mu_{\mathcal{Z}}(Y')$, if $< x, i >$ is any copy of x, then there is $< y, j > \prec < x, i >$, $y \in Y'$. Consider now $U \cup Y'$. As $< x, i >$ is not minimal in $\mathcal{Z}\lceil U \cup Y'$, by smoothness of $\mathcal{Z}$ there must be $< z, k > \prec < x, i >$, $< z, k >$ minimal in $\mathcal{Z}\lceil U \cup Y'$. But all minimal elements of $\mathcal{Z}\lceil U \cup Y'$ must be in $\mathcal{Z}\lceil U$, so there must be $< z, k > \prec < x, i >$, $z \in U$, thus $< x, i >$ is not minimal in $\mathcal{Z}\lceil U$. Thus by $\mu_{\mathcal{Z}}(U) \subseteq \mu(U)$, $\mu_{\mathcal{Z}}(U) \subseteq \mu(U) - U'$, so $\overbrace{\mu_{\mathcal{Z}}(U)} \subseteq \overbrace{\mu(U) - U'} \subset \mu(U)$, *contradiction.*

(μCUM) : Let $\mu(X) \subseteq Y \subseteq X$. Now $\mu_{\mathcal{Z}}(X) \subseteq \overbrace{\mu_{\mathcal{Z}}(X)} = \mu(X)$, so by smoothness of $\mathcal{Z}$ $\mu_{\mathcal{Z}}(Y) = \mu_{\mathcal{Z}}(X)$, thus $\mu(X) = \overbrace{\mu_{\mathcal{Z}}(X)} = \overbrace{\mu_{\mathcal{Z}}(Y)} = \mu(Y)$. $\square$

4.3.2 The ranked case

We recall from [Sch04] Notation 4.1, Definition 4.4, Fact 4.11, Proposition 4.12.

Notation 4.1

(1) A $= B \parallel C$ stands for: $A = B$ or $A = C$ or $A = B \cup C$.

(2) Given $\prec$, $a \perp b$ means: neither $a \prec b$ nor $b \prec a$.

Definition 4.4

The new conditions for the minimal case are:

$(\mu\emptyset)$ $X \neq \emptyset \rightarrow \mu(X) \neq \emptyset$,

$(\mu\emptyset fin)$ $X \neq \emptyset \rightarrow \mu(X) \neq \emptyset$ for finite X,

$(\mu =)$ $X \subseteq Y$, $\mu(Y) \cap X \neq \emptyset \rightarrow \mu(Y) \cap X = \mu(X)$,

$(\mu =')$ $\mu(Y) \cap X \neq \emptyset \rightarrow \mu(Y \cap X) = \mu(Y) \cap X$,

$(\mu \parallel)$ $\mu(X \cup Y) = \mu(X) \parallel \mu(Y)$,

$(\mu\cup)$ $\mu(Y) \cap (X - \mu(X)) \neq \emptyset \rightarrow \mu(X \cup Y) \cap Y = \emptyset$,

$(\mu\cup')$ $\mu(Y) \cap (X - \mu(X)) \neq \emptyset \rightarrow \mu(X \cup Y) = \mu(X)$,

$(\mu \in)$ $a \in X - \mu(X) \rightarrow \exists b \in X. a \notin \mu(\{a, b\})$.

We will use

Fact 4.11

The following properties $(2) - (9)$ hold, provided corresponding closure conditions for the domain $\mathcal{Y}$ are satisfied. We first enumerate these conditions.

For (3), (4), (8): closure under finite unions.

For (2): closure under finite intersections.

For (6) and (7): closure under finite unions, and $\mathcal{Y}$ contains all singletons.

For (5): closure under set difference.

For (9): suffienctly strong conditions - which are satisfied for the set of models definable by propositional theories.

Note that the closure conditions for (5), (6), (9) are quite different, for this reason, (5) alone is not enough.

(1) $(\mu =)$ entails (μPR),

(2) in the presence of $(\mu \subseteq)$, $(\mu =)$ is equivalent to $(\mu =')$,

(3) $(\mu \subseteq)$, $(\mu =) \rightarrow (\mu\cup)$,

(4) $(\mu \subseteq)$, $(\mu\emptyset)$, $(\mu =)$ entail:

(4.1) $(\mu \parallel)$,

(4.2) $(\mu\cup')$,

(4.3) (μCUM),

(5) $(\mu \subseteq) + (\mu \parallel) \rightarrow (\mu =)$,

(6) $(\mu \parallel) + (\mu \in) + (\mu PR) + (\mu \subseteq) \rightarrow (\mu =)$,

(7) $(\mu CUM) + (\mu =) \rightarrow (\mu \in)$,

(8) $(\mu CUM) + (\mu =) + (\mu \subseteq) \rightarrow (\mu \parallel)$,

(9) $(\mu PR) + (\mu CUM) + (\mu \parallel) \rightarrow (\mu =)$.

and

Proposition 4.12

Let $\mathcal{Y} \subseteq \mathcal{P}(U)$ be closed under finite unions, and contain singletons. Then $(\mu \subseteq)$, $(\mu \emptyset fin)$, $(\mu =)$, $(\mu \in)$ characterize ranked structures for which for all finite $X \in \mathcal{Y}$ $X \neq \emptyset \rightarrow \mu_<(X) \neq \emptyset$ hold, i.e. $(\mu \subseteq)$, $(\mu \emptyset fin)$, $(\mu =)$, $(\mu \in)$ hold in such structures for $\mu_<$, and if they hold for some μ, we can find a ranked relation $<$ on U s.t. $\mu = \mu_<$.

Note that wlog. we may assume that the structure contains no copies.

We give now an easy version of representation results for ranked structures without definability preservation.

Notation 4.2

We abbreviate $\mu(\{x, y\})$ by $\mu(x, y)$ etc.

Fact 4.13

Let the domain contain singletons and be closed under $(\cup)$.

Let for $\mu : \mathcal{Y} \rightarrow \mathcal{Y}$ hold:

$(\mu =)$ for finite sets, $(\mu \in)$, $(\mu PR3)$, $(\mu \emptyset fin)$.

Then the following properties hold for μ_3 as defined in Definition 4.3:

(1) $\mu_3(X) \subseteq \mu(X)$,

(2) for finite X, $\mu(X) = \mu_3(X)$,

(3) $(\mu \subseteq)$,

(4) (μPR),

(5) $(\mu \emptyset fin)$,

(6) $(\mu =)$,

(7) $(\mu \in)$,

(8) $\mu(X) = \overbrace{\mu_3(X)}$.

Proof:

(1) Suppose not, so $x \in \mu_3(X)$, $x \in X - \mu(X)$, so by $(\mu \in)$ for μ, there is $y \in X$, $x \notin \mu(x,y)$, *contradiction.*

(2) By $(\mu PR3)$ for μ and (1), for finite U $\mu(U) = \mu_3(U)$.

(3) $(\mu \subseteq)$ is trivial for μ_3.

(4) Let $X \subseteq Y$, $x \in \mu_3(Y) \cap X$, suppose $x \in X - \mu_3(X)$, so there is $y \in X \subseteq Y$, $x \notin \mu(x,y)$, so $x \notin \mu_3(Y)$.

(5) $(\mu\emptyset fin)$ for μ_3 follows from $(\mu\emptyset fin)$ for μ and (2).

(6) Let $X \subseteq Y$, $y \in \mu_3(Y) \cap X$, $x \in \mu_3(X)$, we have to show $x \in \mu_3(Y)$. By (4), $y \in \mu_3(X)$. Suppose $x \notin \mu_3(Y)$. So there is $z \in Y.x \notin \mu(x,z)$. As $y \in \mu_3(Y)$, $y \in \mu(y,z)$. As $x \in \mu_3(X)$, $x \in \mu(x,y)$, as $y \in \mu_3(X)$, $y \in \mu(x,y)$. Consider $\{x,y,z\}$. Suppose $y \notin \mu(x,y,z)$, then by $(\mu \in)$ for μ, $y \notin \mu(x,y)$ or $y \notin \mu(y,z)$, *contradiction.* Thus $y \in \mu(x,y,z) \cap \mu(x,y)$. As $x \in \mu(x,y)$, and $(\mu =)$ for μ and finite sets, $x \in \mu(x,y,z)$. Recall that $x \notin \mu(x,z)$. But for finite sets $\mu = \mu_3$, and by (4) (μPR) holds for μ_3, so it holds for μ and finite sets. *contradiction*

(7) Let $x \in X - \mu_3(X)$, so there is $y \in X.x \notin \mu(x,y) = \mu_3(x,y)$.

(8) As $\mu(X) \in \mathcal{Y}$, and $\mu_3(X) \subseteq \mu(X)$, $\overbrace{\mu_3(X)} \subseteq \mu(X)$, so by $(\mu PR3)$ $\overbrace{\mu_3(X)} = \mu(X)$.

□

Fact 4.14

If $\mathcal{Z}$ is ranked, and we define $\mu(X) := \overbrace{\mu_{\mathcal{Z}}(X)}$, and $\mathcal{Z}$ has no copies, then the following hold:

(1) $\mu_{\mathcal{Z}}(X) = \{x \in X : \forall y \in X.x \in \mu(x,y)\}$, so $\mu_{\mathcal{Z}}(X) = \mu_3(X)$ for $X \in \mathcal{Y}$,

(2) $\mu(X) = \mu_{\mathcal{Z}}(X)$ for finite X,

(3) $(\mu =)$ for finite sets for μ,

(4) $(\mu \in)$ for μ,

(5) $(\mu\emptyset fin)$ for μ,

(6) $(\mu PR3)$ for μ.

Proof:

(1) holds for ranked structures.

(2) and (6) are trivial. (3) and (5) hold for $\mu_{\mathcal{Z}}$, so by (2) for μ.

(4) If $x \notin \mu(X)$, then $x \notin \mu_{\mathcal{Z}}(X)$, $(\mu \in)$ holds for $\mu_{\mathcal{Z}}$, so there is $y \in X$ s.t. $x \notin \mu_{\mathcal{Z}}(x,y) = \mu(x,y)$ by (2).

$\square$

We summarize:

Proposition 4.15

Let Z be an arbitrary set, $\mathcal{Y} \subseteq \mathcal{P}(Z)$, $\mu : \mathcal{Y} \to \mathcal{Y}$, $\mathcal{Y}$ closed under arbitrary intersections and finite unions, contain singletons, and $\emptyset, Z \in \mathcal{Y}$, and let $\widehat{\ }$ be defined wrt. $\mathcal{Y}$.

(a) If μ satisfies $(\mu =)$ for finite sets, $(\mu \in)$, $(\mu PR3)$, $(\mu\emptyset fin)$, then there is a ranked preferential structure $\mathcal{Z}$ without copies over Z s.t. for all $U \in \mathcal{Y}$ $\mu(U) = \widehat{\mu_{\mathcal{Z}}(U)}$.

(b) If $\mathcal{Z}$ is a ranked preferential structure over Z without copies and $\mu : \mathcal{Y} \to \mathcal{Y}$ s.t. for all $U \in \mathcal{Y}$ $\mu(U) = \widehat{\mu_{\mathcal{Z}}(U)}$, then μ satisfies $(\mu =)$ for finite sets, $(\mu \in)$, $(\mu PR3)$, $(\mu\emptyset fin)$.

Proof:

(a) Let μ satisfy $(\mu =)$ for finite sets, $(\mu \in)$, $(\mu PR3)$, $(\mu\emptyset fin)$, then μ_3 as defined in Definition 4.3 satisfies properties $(\mu \subseteq)$, $(\mu\emptyset fin)$, $(\mu =)$, $(\mu \in)$ by Fact 4.13. Thus, by Proposition 4.12, there is a transitive structure $\mathcal{Z}$ over Z s.t. $\mu_3 = \mu_{\mathcal{Z}}$, but by Fact 4.13 (8) $\mu(U) = \widehat{\mu_3(U)} = \widehat{\mu_{\mathcal{Z}}(U)}$ for $U \in \mathcal{Y}$.

(b) This was shown in Fact 4.14.

$\square$

5 OUTLOOK: PATCHY DOMAINS AND WEAK DIAGNOSTIC INSTRUMENTS

We have seen in these pages two kinds of problems with repercussions on representation questions:

(1) Lack of closure of the domain, in particular under finite union.

(2) Lack of definability preservation.

We summarize here the problem:

(1) If the domain is not closed under suitable operations, we may be forced to replace simple conditions like cumulativity by more complicated ones.

(2) If the operation is not definability preserving, we may be foced to admit exceptions.

In realistic situations, first, both problems may very well occur at the same time. Second, it is not evident that observable situations coincide with logically describable situations. Moreover, it is not all clear that we can perform an "experiment" with all formulas or theories which are logically describable.

Thus, we have three elements:

(1) A domain which may be quite patchy.

(2) Limited possibilities to carry out "experiments", i.e. a limited number of input situations, which may be only a subset of what is logically possible.

(3) A limited number of observable results.

We have a shaggy situation, can do some experiments, and then might see the result only through a rather rough grid.

The author has intentionally chosen here the language of scientific experiments, as our situation seems close to the latter problem. So, perhaps, both sides can learn from each other.

At the moment, it seems difficult to obtain general results, as for instance, the different diagnostic instruments for smooth and general preferential situations (μ_0 and μ_2) show. As long as we have no further information about observability, for instance inconsistency need not always be observable, as $\emptyset \neq \widehat{\ \emptyset\ }$ may very well be possible. But the problem might be too general, and is amenable only for restricted cases - some of which were solved above.

Note that we have a similar situation in preferential structures, copies: We cannot observe copies directly, they are not describable in our language, we only see their effects.

It seems rather plausible that similar problems do not only occur in the context of non-monotonic reasoning. It seems safe to conjecture that there are existing systems and approaches which do not take into account the problems we have seen here, i.e. work fine under ideal situations, but are not adapted to their particular domain or observability problems.

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